153. *An Integral of the Denjoy Type*

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(Comm. by Kinjirô KUNUGI, M.J.A., Nov. 12, 1964)

1. Introduction. In the present paper, we shall consider an integral of the Denjoy type whose indefinite integral is approximately continuous. H. W. Ellis [2] has introduced the GM-integral descriptively. Defining our integral we use his method, which is essentially based on the procedure introduced by S. Saks [3] and W. L. C. Sargent [4]. It will be proved that our integral is more general than Burkill's approximately continuous Perron integral [1].

2. A finite function $f(x)$ is said to be $\underline{AC}$ on a set E if to each positive number ε, there exists a number $\delta>0$ such that

$$\sum\{f(b_k)-f(a_k)\}>-\varepsilon$$

for all finite non-overlapping sequences of intervals $\{(a_k, b_k)\}$ with end points on E and such that $\sum(b_k-a_k)<\delta$. There is a corresponding definition $\overline{AC}$ on E. If the set E is the sum of a countable number of sets E_k on each of which $f(x)$ is $\underline{AC}$ then $f(x)$ is termed $\underline{ACG}$ on E. If the sets E_k are assumed to be closed, then $f(x)$ is said to be $(\underline{ACG})$ on E. Similarly we can define $\overline{ACG}$ and $(\overline{ACG})$ on E. A function is said to be (ACG) on E if it is both $(\underline{ACG})$ and $(\overline{ACG})$ on E.

Lemma 1. *If $F(x)$ is $\underline{AC}$ and $AD\ F(x)\geqq 0$ almost everywhere on $[a, b]$ then $F(x)$ is non-decreasing on $[a, b]$.*

Proof. Since $F(x)$ is $\underline{AC}$ on $[a, b]$, for a given $\varepsilon>0$ we can find $\delta>0$ such that

$$\sum\{F(b_k)-F(a_k)\}>-\varepsilon$$

for all finite non-overlapping sequences of intervals $\{(a_k, b_k)\}$ with $\sum(b_k-a_k)<\delta$.

If we put $E=\{x\colon AD\ F(x)\geqq 0\}$ then $|E|=b-a$. For any $x\in E$ there exists a positive sequence h_k such that

$$\frac{F(x+h_k)-F(x)}{h_k}>-\varepsilon,\ (k=1, 2, \cdots)$$

and $h_k\to 0$. Let M be the family of the sets of closed intervals $[x, x+h_k]$ $(k=1, 2, \cdots)$ for all $x\in E$, then E is covered by M in the sense of Vitali. Hence we can select a finite sequence of non-overlapping intervals in M

$$[x_1, x_1'],\ [x_2, x_2'],\ \cdots,\ [x_m, x_m']$$

such that

$$|E-\bigcup_{k=1}^{m}[x_k, x'_k]|<\delta$$

or

$$|[a, b]-\bigcup_{k=1}^{m}[x_k, x'_k]|<\delta.$$

Since $F(x)$ is $\underline{AC}$ on $[a, b]$ and total length of the sequence of closed intervals

$$[a, x_1], [x'_1, x_2], \cdots, [x'_m, b]$$

is less than δ, we have

$$(1) \qquad \sum_{k=1}^{m+1}\{F(x_k)-F(x'_{k-1})\}>-\varepsilon,$$

where $x'_0=a$, $x_{m+1}=b$. On the other hand, it holds that

$$(2) \qquad F(x'_k)-F(x_k)>-\varepsilon(x'_k-x_k) \quad (k=1, 2, \cdots m).$$

Hence it follows from (1) and (2)

$$F(b)-F(a)>-\varepsilon-\varepsilon(b-a),$$

which implies $F(b)\geqq F(a)$.

Lemma 2. *If $F(x)$ is approximately continuous on $[a, b]$ and is non-decreasing for $a<x<b$ then $F(x)$ is non-decreasing on $[a, b]$.*

Proof. Let $a<\alpha<\beta<b$. Then $F(x)$ is non-decreasing on $[\alpha, \beta]$. Hence it is sufficient to prove that $F(x)$ is non-decreasing in the neighbourhood of points a and b. Suppose that there exists a point x_0 with $F(a)>F(x_0)$. Since $F(x)$ is approximately continuous at a, we can find a point a' sufficiently near to a such that

$$|F(a)-F(a')|<F(a)-F(x_0) \quad (a<a'<x_0).$$

Hence $F(x_0)<F(a')$ which leads to a contradiction. Similarly we can prove that $F(x)$ is non-decreasing in the neighbourhood of the point b.

Lemma 3. *Let $F(x)$ be approximately continuous and $(\underline{ACG})$ on $[a, b]$ and let $AD\ F(x)\geqq 0$ almost everywhere on $[a, b]$. If P is a perfect set on $[a, b]$ with $F(x)$ non-decreasing on the complementary intervals $\{(a_k, b_k)\}$, then there is an interval (l, m) contaning points of P with $F(x)$ non-decreasing on (l, m).*

Proof. Since $F(x)$ is $(\underline{ACG})$ on $[a, b]$, $[a, b]$ is the sum of a countable number of closed sets E_k on each of which $F(x)$ is $\underline{AC}$. We can write $P=\sum P\cdot E_k$, and therefore, by Baire's cathegory theorem ([3], p. 54), there exists an interval (l, m) and a natural number k_0 such that $P\cdot(l, m)\subset P\cdot E_{k_0}$. Hence $F(x)$ is $\underline{AC}$ on $P\cdot(l, m)$.

If we put

$$F_1(x)=F(x) \qquad \text{on } P\cdot(l, m),$$
$$=F(a_k)+\frac{x-a_k}{b_k-a_k}\{F(b_k)-F(a_k)\} \text{ for } x\in[a_k, b_k],$$

then it is shown that $F_1(x)$ is $\underline{AC}$ on (l, m). Since $F(x)$ is approximately continuous and non-decreasing on $a_k<x<b_k$, it follows from

Lemma 2 that $F(x)$ is non-decreasing on $[a_k, b_k]$. Hence $F_1(x)$ is, by the definition, non-decreasing on $[a_k, b_k]$.

Since $P\cdot(l, m)$ is measurable, almost all points of $P\cdot(l,m)$ are points of density of $P\cdot(l, m)$. Therefore, by the assumption $AD\ F(x)\geqq 0$, we have $AD\ F_1(x)\geqq 0$ for almost all points of $P\cdot(l, m)$. Hence $AD\ F_1(x)\geqq 0$ almost everywhere on (l, m). It follows from Lemma 1 that $F_1(x)$ is non-decreasing on (l, m), and therefore $F(x)$ is non-decreasing on $P\cdot(l, m)$.

Theorem 1. *If $F(x)$ is approximately continuous, $(\underline{ACG})$ on $[a, b]$ and $AD\ F(x)\geqq 0$ a.e. then $F(x)$ is non-decreasing on $[a, b]$.*

Proof. Let E be the set of points of $[a, b]$ throughout no neighbourhood of which $F(x)$ is non-decreasing. It is clear that E is closed. If we assume that E has an isolated point x_0, then there exists an interval (p, q) $(p<x_0<q)$ which contains no points of E except x_0. Since $F(x)$ is non-decreasing on (p, x_0) and (x_0, q), it follows from Lemma 2 that $F(x)$ is non-decreasing on $[p, x_0]$ and $[x_0, q]$. Hence $F(x)$ is so on $[p, q]$ which leads to a contradiction. Therefore E is perfect or empty.

Suppose that E is not empty. Let $\{(a_k, b_k)\}$ be the sequence of complementary intervals of the perfect set E. Then $F(x)$ is non-decreasing on (a_k, b_k). It follows from the assumptions and Lemma 3 that there exists an interval (l, m) containing points of E such that $F(x)$ is non-decreasing on (l, m). This contradicts the definition of E, which proves the theorem.

3. Let $f(x)$ be a function defined on $[a, b]$ and suppose there exists a function $F(x)$ such that

(i) $F(x)$ is approximately continuous on $[a, b]$,

(ii) $F(x)$ is (ACG) on $[a, b]$,

(iii) $AD\ F(x)=f(x)$ a.e.,

then $f(x)$ is said to be integrable on $[a, b]$ in the approximately continuous Denjoy sense or AD-integrable. We then say that the function $F(x)$ is an indefinite AD-integral of $f(x)$. Its increment $F(b)-F(a)$ is called definite AD-integral of $f(x)$ on $[a, b]$ and is denoted by $(AD)\int_a^b f(t)dt$.

It follows from Theorem 1 that indefinite AD-integral of $f(x)$ is uniquely determined except an additive constant.

We state some elementary properties which may be proved directly from the definition of the AD-integral. (i) If $f(x)$ is AD-integrable on $[a, b]$ and $f(x)=g(x)$ a.e., then $g(x)$ is also AD-integrable and

$$(AD)\int_a^b f(t)dt=(AD)\int_a^b g(t)dt.$$

(ii) If $f(x)$ and $g(x)$ are both AD-integrable on $[a, b]$, then $\alpha f(x)+\beta g(x)$ is AD-integrable and

$$(AD)\int_a^b(\alpha f+\beta g)dt=\alpha(AD)\int_a^b f(t)dt+\beta(AD)\int_a^b g(t)dt.$$

Next we shall show that the AD-integral is more general than Burkill's approximately continuous Perron integral (AP-integral).

Theorem 2. *If $f(x)$ is AP-integrable on $[a, b]$ then $f(x)$ is also AD-integrable and*

$$(AD)\int_a^b f(t)dt=(AP)\int_a^b f(t)dt.$$

Proof. If we put

$$F(x)=(AP)\int_a^x f(t)dt$$

then it is known ([1], p. 276) that $F(x)$ is approximately continuous on $[a, b]$ and $AD\ F(x)=f(x)$ a.e.

Since $f(x)$ is AP-integrable, there exists a sequence of upper functions $\{U_k(x)\}$ and a sequence of lower functions $\{L_k(x)\}$ such that

$$\lim_{k\to\infty} U_k(b)=\lim_{k\to\infty} L_k(b)=F(b).$$

The functions $U_k(x)-F(x)$ and $F(x)-L_k(x)$ are non-decreasing ([1], p. 273), so that we have, for $x\in[a, b]$,

(1) $$\lim_{k\to\infty} U_k(x)=\lim_{k\to\infty} L_k(x)=F(x).$$

Since $\underline{AD}\ U_k(x)>-\infty\ [\overline{AD}\ L_k(x)<+\infty]$ and since $U_k(x)[L_k(x)]$ is approximately continuous, it follows Ridder's theorem ([5], p. 153, footnote) that $U_k(x)$ $[L_k(x)]$ is $\underline{(ACG)}$ $[\overline{(ACG)}]$ on $[a, b]$. Hence the interval $[a, b]$ is expressible as the sum of a countable number of closed sets E_k, $[a, b]=\bigcup E_k$, such that any U_k is $\underline{AC}$ on any E_k and at the same time any L_k is $\overline{AC}$ on any E_k.

Next we shall show that $F(x)$ is AC on E_k. For this purpose we shall prove that $F(x)$ is both $\underline{AC}$ and $\overline{AC}$ on E_k.

Suppose that $F(x)$ is not $\underline{AC}$ on E_k. Then there exists an $\varepsilon>0$ such that for any small $\delta>0$ we can find non-overlapping intervals (a_ν, b_ν) with end points on E_k satisfying $\sum(b_\nu-a_\nu)<\delta$ but

(2) $$\sum\{F(b_\nu)-F(a_\nu)\}\leqq-\varepsilon.$$

Since we can find a natural number p, by (1), such that

$$U_p(x)-F(x)<\frac{\varepsilon}{2},$$

and since $U_p(x)-F(x)$ is non-decreasing on $[a, b]$ we have

(3) $$\begin{aligned}&\sum\{U_p(b_\nu)-U_p(a_\nu)\}-\sum\{F(b_\nu)-F(a_\nu)\}\\&=\sum[(U_p(b_\nu)-F(b_\nu))-(U_p(a_\nu)-F(a_\nu))]\\&\leqq U_p(b)-F(b)<\frac{\varepsilon}{2}.\end{aligned}$$

It follows from (2) and (3) that

$$\sum\{U_p(b_\nu)-U_p(a_\nu)\}<\sum\{F(b_\nu)-F(a_\nu)\}+\frac{\varepsilon}{2}$$
$$\leqq -\frac{\varepsilon}{2}.$$

This contradicts the fact that $U_p(x)$ is $\underline{AC}$ on E_k, and therefore $F(x)$ is $\underline{AC}$ on E_k.

Similarly we can prove that $F(x)$ is $\overline{AC}$ on E_k. Thus $F(x)$ is AC on each closed set E_k, and also (ACG) on $[a, b]$. Since $F(x)$ is approximately continuous and $AD\ F(x)=f(x)$ a.e. it follows that $f(x)$ is AD-integrable on $[a, b]$ and that

$$(AD)\int_a^b f(t)dt=(AP)\int_a^b f(t)dt.$$

References

[1] J. C. Burkill: The approximately continuous Perron integral. Math. Zeit., **34**, 270-278 (1931).
[2] H. W. Ellis: Mean continuous integrals. Canadian Jour. Math., **1**, 113-124 (1949).
[3] S. Saks: Theory of the Integral. Warsaw (1937).
[4] W. L. C. Sargent: A descriptive definition of Cesàro-Perron integrals. Proc. London Math. Soc., **47**, 212-247 (1941).
[5] J. Ridder: Ueber die gegenseitigen Beziehungen vershieder approximativ stetiger Denjoy-Perron-Integrale. Fund. Math., **22**, 136-162 (1934).