# 150. A Note on Riemann's Period Relation

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1. Let $W$ be a Riemann surface of infinite genus and $\mathfrak{J}$ the ideal boundary of $W$. First we consider the following classes of dividing cycles on $W$.

DEFINITION 1. *A dividing cycle $C$ on $W$ belongs to the class $\mathfrak{D}_h$ of dividing cycles of order at most $h$ when, for $h>1$, $C$ can be written as $C=\sum_{k=1}^{K}\sigma_k$ with some $K\leqslant h$ where $\sigma_k$ is a closed curve, and for any $i\leqslant K$ $\sigma_1\cdots\sigma_{i-1}, \sigma_{i+1}\cdots\sigma_K$ are homologously independent mod $\mathfrak{J}$. For $h=1$ $\mathfrak{D}_1$ is the class of connected dividing curves.*

DEFINITION 2. *A dividing cycle on $W$ belongs to the class $\mathfrak{D}'_h$ $(\subset\mathfrak{D}_h)$ of dividing cycles of order $h$, when it is written as $K=h$ in* Def. 1, *i.e.* $\mathfrak{D}'_h=\mathfrak{D}_h-\mathfrak{D}_{h-1}$.

DEFINITION 3. *An exhaustion of $W$ by regular regions $(F_n)$ belongs to the class $\mathcal{E}_h$ of semi canonical exhaustions of at most order $h$, when it satisfies the following conditions:*

(A) (i) *It is an exhaustion in Noshiro's sense, (cf. [6], p. 50).**)

(ii) *Denoting canonical partition $Q$ of the set of the contours of $F_n$ (cf. [3]) by $Q(\partial F_n)=\sum_{i=1}^{m(n)}\Gamma_n^i$, $(\Gamma_n^i\in\mathfrak{D}_h$, $\Gamma_n^i=\sum_{k=1}^{K(n,i)}\sigma_{nk}^i$, and $\sigma_{nk}^i$ is a closed contour) there exist at least one $\Gamma_n^i$ such that $\Gamma_n^i\in\mathfrak{D}'_h$.*

(iii) *$\Gamma_n^i\sim\sum_j\Gamma_{n+1}^{ij}$ $(\Gamma_n^i, \Gamma_{n+1}^{ij}\in\mathfrak{D}_h)$ being inner and outer boundary of a component $F_n^i$ of $F_{n+1}-\bar{F}_n$, there is only one component of $F_{n+2}-\bar{F}_{n+1}$ which is adjoined to $F_n^i$ along each $\Gamma_{n+1}^{ij}$.*

2. By using Lemma 5 in [2], slit method in [7], and Noshiro's graph in [6], we can prove easily the following

LEMMA 1. *For $h\geqslant 1$, $\mathcal{E}_h\neq\phi$.*

Let $D_{nk}^i$ be an annulus which satisfies the conditions:

(B) (i) $D_{nk}^i$ includes $\sigma_{nk}^i$ and $\overline{D_{nk}^i}$ is a closed annulus contained in $F_{n+1}-\overline{F_{n-1}}$.

(ii) $\overline{D_{nk}^i}\cap\overline{D_{m\beta}^\alpha}=\phi$ if $n\neq m$ or $i\neq\alpha$ or $k\neq\beta$.

We put $D_n^i=\sum_{k=1}^{K(n,i)}D_{nk}^i$, $D_n=\sum_{i=1}^{m(n)}D_n^i$, $\partial D_n=\beta_n-\alpha_n$. Let $M_{nk}^i$, $M_n^i$, and $M_n$ be the moduli of $D_{nk}^i$, $D_n^i$, and $D_n$ respectively. We consider a harmonic function $u_n$ in $D_n$ which vanishes on $\alpha_n$ and is equal to $M_n$ on $\beta_n$, and the conjugate $v_n$ of $u_n$ satisfies $\int_{\beta_n}dv_n=2\pi$. Putting $u+iv$

*) [$p$] means the $p$-th paper in References which are shown at the end of this paper.

$=u_n+iv_n+\sum_{j=1}^{n-1} M_j$ in each $D_n$, we can map $D=\sum_{n=1}^{\infty} D_n$ by $u+iv$ onto a strip domain $0<u<R=\sum_{n=1}^{\infty} M_n$, $0<v<2\pi$ (cf. [4]). For $\sum_{j=1}^{n-1} M_j<r\leqslant\sum_{j=1}^{n} M_j$, we denote the level curve $u=r$ by $\Gamma_r=\sum_{i=1}^{m(n)} \Gamma_r^i$ $(\Gamma_r^i\in\mathfrak{D}_h)$ and put

$$L_i(r)=\Big(\int_{\Gamma_r^i} |\omega_1|+|\omega_2|\Big)^2,\quad L(r)=\sum_{i=1}^{m(n)} L_i(r),$$

where $\omega_1, \omega_2\in\Gamma_h(W)$. Then we can obtain the following lemma by the same way as Lemma 1 in [4] was established.

LEMMA 2. *If $\sum_{n=1}^{\infty} \min_i \min_k M_{nk}^i$ is divergent, there exists a sequence $\{\gamma_n\}$ of level curves $u=r_n$ tending to $\mathfrak{J}$ such that $\lim_{n\to\infty} L(r_n)=0$.*

REMARK. If we choose a suitable subsequence of $\{\gamma_n\}$ we can assume that $(F'_{nj}|\partial F'_{nj}=\gamma_{nj})$ belongs to $\mathcal{E}_h$.

DEFINITION 4. *Such an exhaustion as mentioned above is called an exhaustion associated with $(F_n)$ and $\omega_1$, $\omega_2$.*

3. We suppose $(F_n)\in\mathcal{E}_h$ and put $F_{n+2}-F_n=\sum_{i=1}^{m(n)} E_n^i$ and $F_{n+1}-\bar{F}_n=\sum_{i=1}^{m(n)} F_n^i$. Next we are going to construct some family of curves on each component $E_n^i$ and $F_n^i$. Hereafter we put $E_n^i=E$, $F_n^i=F$ and

$$Q'(\partial F)=\sum_{j=1}^{N} \Gamma^j-\alpha,\quad \Gamma^j=\sum_{\nu=1}^{K_j} \sigma_\nu^j\quad (\alpha, \Gamma^j\in\mathfrak{D}_h,\ K_j\leqslant h). \tag{3.1}$$

We fix a point $P_\nu^j$ on $\sigma_\nu^j$ and connect $P_\nu^j$ and $P_{\nu+1}^j$ by two analytic curves $C_\nu'^j$, $C_\nu''^j$, $j=1, \cdots N$, $\nu=1, \cdots K_j-1$, which satisfy the following conditions (cf. condition (B)):

(C) (i) $C_\nu'^j$ is in $F-(\sum_{k\neq\nu,\nu+1} \bar{D}_k^j\cap\bar{F})$, and $C_\nu''^j$ is in $E-F-\sum_{k\neq\nu,\nu+1} \bar{D}_k^j\cap(E-\bar{F})$.

(ii) $C_\nu'^j+C_\nu''^j=C_\nu^j$ is a closed analytic curve with the orientation from $P_\nu^j$ to $P_{\nu+1}^j$ in $F$.

(iii) $C_\nu^j\cap C_\beta^\alpha=\phi$, for $\alpha\neq j$ or $\beta\neq\nu$.

We cut $E$ along $C_\nu^j$ (for fixed $j$ and $\nu$), and denote by $G$ the cut surface and by $\tilde{C}_\nu^j$ the new boundary that corresponds to the left side of $C_\nu^j$ with respect to its orientation, and by $\tilde{C}_\nu'^j$ the other side. Let $u_\nu^j$ be a harmonic function on $G$ which vanishes on $\tilde{C}_\nu^j$, is equal to 1 on $\tilde{C}_\nu'^j$, and normal derivative $\dfrac{\partial u_\nu^j}{\partial n}=0$ on other boundaries, then we can get (cf. [3])

LEMMA 3. $\displaystyle\int_{\tilde{C}_\nu^j} du_\nu^{*j}=\int_{u_\nu^j=\rho} du_\nu^{*j}=\lambda_G(\tilde{C}_\nu^j, \tilde{C}_\nu'^j)=\lambda_E(C_\nu^j),$

*where $du_\nu^{*j}$ is a conjugate harmonic differential of $du_\nu^j$, $(\tilde{C}_\nu^j, \tilde{C}_\nu'^j)$ a family of curves that divide $\tilde{C}_\nu^j$ from $\tilde{C}_\nu'^j$ on $G$, $(C_\nu^j)$ a family of curves that are homologous to $C_\nu^j$ in $E$, and $\lambda_G(\tilde{C}_\nu^j, \tilde{C}_\nu'^j)$ is the extremal length of the family $(\tilde{C}_\nu^j, \tilde{C}_\nu'^j)$ on $G$.*

With these preparations we consider the integral of $|\omega_1|$, $|\omega_2|$ along the curve $C_\nu^j(\rho)|u_\nu^j=\rho$, where $\omega_1, \omega_2 \in \Gamma_h(W)$. Since $u_\nu^j+iu_\nu^{*j}$ is considered as a uniformizer on $\bar{G}$, we can put $\omega_k=a_k du_\nu^j+b_k du_\nu^{*j}$.

Then we get from Lemma 3 by the Schwarz's inequality

$$(l_\nu^j(\rho))^2=\Big(\int_{C_\nu^j(\rho)} |\omega_1|+|\omega_2|\Big)^2 \leqslant 2\lambda_E(C_\nu^j)\int_{C_\nu^j(\rho)} (|b_1|^2+|b_2|^2)du_\nu^{*j}.$$

Hence if $\lambda_E(C_\nu^j)\leqslant A$ for $j=1, \cdots N$, $\nu=1, \cdots K_j-1$, we get

$$\int_0^1 \sum_{j=1}^{N}\sum_{\nu=1}^{Kj-1}(l_\nu^j(\rho))^2 d\rho=\int_0^1 M'(\rho)d\rho\leqslant 2ANh(||\omega_1||_E^2+||\omega_2||_E^2). \tag{3.2}$$

We denote $A, N, M'(\rho)$ with respect to a component $E_n^i$ by $A_n^i$, $N_n^i$, and $M_n'^i(\rho)$. If $A_n^i<A_n$, $N_n^i<N_n$ for all $i$, we get

$$\int_0^1 \sum_{i=1}^{m(n)} M_n'^i(\rho)d\rho=\int_0^1 M_n'(\rho)d\rho\leqslant 2A_nN_nh(||\omega_1||^2_{F_{n+2}-\bar{F}_n}+||\omega_2||^2_{F_{n+2}-\bar{F}_n}). \tag{3.3}$$

DEFINITION 5. *Above mentioned curve $C_\nu^j(\rho)$ is called a $\rho$-cycle in $E$, whose orientation is coherent to that of $C_\nu^j$. $(C_\nu^j)$ is called a C-cycle family in $E$.*

Let $(W_\mu)$ be an exhaustion associated with $(F_{3n+1})$ and $\omega_1$, $\omega_2$, then we have the following statements:

(H) For each $\mu$ there corresponds an integer $n_\mu$ uniquely such that each component of $\partial W_\mu$ is homologous to a corresponding component of $\partial F_{3n_\mu+1}$, and when, for simplicity, we put $F_{3n_\mu}=F'_{3\mu}$, $F_{3n_\mu+1}=F'_{3\mu+1}$, $F_{3n_\mu+2}=F'_{3\mu+2}$ we get $F'_{3\mu}\subset W_\mu\subset F'_{3\mu+2}$, $F'_{3\mu-1}\subset F'_{3\mu}$ (cf. Defs. 3 and 4).

(J) If $\overline{\lim}\, A_nN_n<$constant $K$, for large $\mu$ we get from (3.3)

$$\int_0^1 \sum_{i=1}^{m(3\mu)} M_{3\mu}'^i(\rho)d\rho=\int_0^1 M_\mu'(\rho)d\rho\leqslant 4Kh(||\omega_1||^2_{F'_{3\mu+2}-\bar{F}'_{3\mu}}+||\omega_2||^2_{F'_{3\mu+2}-\bar{F}'_{3\mu}}),$$

where $m(3\mu)=m(3n_\mu)$.

Hence we get

$$\lim_{\mu\to\infty}\int_0^1 M_\mu'(\rho)d\rho=0, \tag{3.4}$$

so there is a subsequence $\{\mu_k\}$ such that for almost everywhere on $[0,1]$ (cf. [8])

$$\lim_{\mu_k\to\infty} M_{\mu_k}'(\rho)=0. \tag{3.5}$$

We will make a new exhaustion from $(W_\mu)$ which will be called the special canonical exhaustion associated with $(F_{3n+1})$ and $\omega_1, \omega_2\in\Gamma_h(W)$.

Let $(W_\mu)$ be an exhaustion which satisfies (3.5), then for fixed $\mu$ we can choose a positive number $\delta_\mu$ such that for all values of $\rho$ $(0<\rho<\delta_\mu)$ each $\rho$-cycle $C_\nu^j(\rho)$ in each component $E=E_\mu^i$ of $F'_{3\mu+2}-\overline{F'_{3\mu}}$ is disjoint with $\partial E$, $\overline{D_k^g}$ $g\neq j$ or $k\neq\nu$, $\nu+1$ and $C_\beta^\alpha(\rho)$ $\alpha\neq j$ or $\beta\neq\nu$ (cf. (B), (C)). Therefore for a fixed $\mu$ and $\rho_\mu$ $(0<\rho_\mu<\delta_\mu)$, $C_\nu^j(\rho_\mu)\cap F$ includes a connected simple curve $S_\nu^j(\rho_\mu)$ which satisfies $S_\nu^j(\rho_\mu)\cap\sigma_\nu'^j\neq\phi$ and $S_\nu^j(\rho_\mu)\cap\sigma_{\nu+1}'^j\neq\phi$, where $F=F_\mu^i$ is a component of $W_\mu-\overline{F'_{3\mu}}$ such that $E\cap F\neq\phi$, and $Q'(\partial F)=\sum^{N}\Gamma'^{j}-\alpha$, $\Gamma'^{j}=\sum^{Kj}\sigma'^{j}_{\nu}$ (cf. (3.1)).

DEFINITION 6. *$S^{j}_{\nu}(\rho_{\mu})$ is called a $\rho_{\mu}$-half cycle in $F$ with respect to $E$.*

From our construction $S^{j}_{\nu}(\rho_{\mu})$ has following properties:

(P) $S^{j}_{\nu}(\rho_{\mu})$ does not intersect each other, $\nu=1,2,\cdots K_{j}-1$, $j=1,2,\cdots N$.

(Q) Cutting $F$ along $S^{j}_{\nu}(\rho_{\mu})$, $\nu=1,\cdots K_{j}-1$, $j=1,\cdots N$, and denoting the cut surface by $G''$, we get $Q'(\partial G'')=\sum\beta'-\alpha$, where $\alpha\in\mathfrak{D}_{h}$ and $\beta'\in\mathfrak{D}_{1}$ is a connected piecewise analytic curve with a finite number of corners.

(R) $$M(\rho_{\mu})=\sum_{j}^{N}\sum_{\nu=1}^{Kj-1}\Big(\int_{S^{j}_{\nu}(\rho_{\mu})}|\omega_{1}|+|\omega_{2}|\Big)^{2}\leqslant 4h^{2}M'(\rho_{\mu}).$$

Denoting $M(\rho_{\mu})$ with respect to $F^{i}_{\mu}$ by $M^{i}_{\mu}(\rho_{\mu})$, we get

$$M_{\mu}(\rho_{\mu})=\sum_{i=1}^{m(3\mu)}M^{i}_{\mu}(\rho_{\mu})\leqslant 4h^{2}M'_{\mu}(\rho_{\mu}) \qquad \text{(cf. (3.5))}.$$

Therefore by the diagonal method

$$\lim M_{\mu}(\rho_{\mu})=0. \tag{3.6}$$

We collect the results in the following Lemma 4.

LEMMA 4. *Let $W$ be an open Riemann surface of infinite genus and $(F_{n})$ be an exhaustion which belongs to $\mathcal{E}_{h}$, and for each $n$ we put*

$$\left.\begin{array}{l} N^{i}_{n}<N_{n}, \text{ where } \; Q(\partial F_{n})=\sum_{i=1}^{(m)n}\Gamma^{i}_{n}, \quad Q(\partial F_{n+1})=\sum_{i=1}^{m(n)}\sum_{j=1}^{N^{i}_{n}}\Gamma^{ij}_{n+1} \\ \Gamma^{i}_{n}\sim\sum\Gamma^{ij}_{n+1}\in\mathfrak{D}_{h} \quad \text{absolutely} \end{array}\right\}. \tag{3.7}$$

*If we assume that*

(i) *$\sum\min_{i}\min_{k}M^{i}_{nk}$ is divergent.*

(ii) *The extremal length of C-cycles in each component of $F_{n+2}-\overline{F}_{n}$ are less than $A_{n}$, and $\overline{\lim}\,A_{n}N_{n}$ is finite.*

*Then we have the following conclusions:*

(1) *There exists a sequence of the level curves $\{\gamma_{\mu}|u=r_{\mu}\}$ tending to $\mathfrak{J}$ such that $\lim_{\mu\to\infty}L(r_{\mu})=0$, and the exhaustion $(W_{\mu}|\partial W_{\mu}=\gamma_{\mu}\}$ is associated with $(F_{3n+1})\in\mathcal{E}_{h}$ and $\omega_{1},\omega_{2}\in\Gamma_{\text{hse}}(W)$.*

(2) *There exists a sequence of positive numbers $\{\rho_{\mu}\}$ such that for $\omega_{1},\omega_{2}\in\Gamma_{\text{hse}}(W)$ in* (1)

$$\lim_{\mu\to\infty}M_{\mu}(\rho_{\mu})=0. \tag{3.6}$$

We cut $W_{\mu}$ along all $\rho_{\mu}$-half cycles in each component of $W_{\mu}-\overline{F}'_{3\mu}$, then we can get a new canonical exhaustion $(W'_{\mu})$, and each component of $\partial W'_{\mu}$ is a dividing cycle consisting of piecewise analytic curves.

DEFINITION 7. *Above mentioned exhaustion $(W'_{\mu})$ is called the special canonical exhaustion associated with $(F_{3n+1})$ and $\omega_{1},\omega_{2}\in\Gamma_{\text{hse}}(W)$.*

4. Let $W$ be an arbitrary Riemann surface and $(F_{n})$ be an exhaustion of $W$. Then there exists on $F_{n}$ a canonical homology basis

such that $A_1, B_1, A_2, B_2, \cdots A_{k_n}, B_{k_n}$ form a canonical basis mod $\partial F_n$ and $A_i \times B_j = \delta_i^j$, $A_i \times A_j = B_i \times B_j = 0$ (cf. [2]). We denote such a basis by H. B. $(F_n)$.

THEOREM 1. *Let $W$ be a Riemann surface which satisfies the conditions of Lemma 4, then for $\omega_1, \omega_2 \in \Gamma_{\rm hse}(W)$ there exists a special canonical exhaustion $(W'_\mu)$ and an H. B. $(W'_\mu)$ such that Riemann's bilinear relation holds.*

PROOF. At first we shall take $(W'_\mu)$ associated with $(F_{3n+1})$ and $\omega_1, \omega_2 \in \Gamma_{\rm hse}(W)$ (cf. Lemma 4 and Def. 7). Next let $S^{ij}_{\mu\nu}(\rho_\mu)$ be a $\rho_\mu$-half cycle on $\partial W'_\mu$ (cf. Defs. 6, 7, and Lemma 4). Then we can get

$$(\omega_1, \omega_2^*)_{W'_\mu} = \sum_{i=1}^{k_\mu} \left( \int_{A_i} \omega_1 \int_{B_i} \bar{\omega}_2 - \int_{A_i} \bar{\omega}_2 \int_{B_i} \omega_1 \right) + (\omega_1, \omega_2^* - (T_{W'_\mu}\omega_2)^*)_{W'_\mu}$$

where $a_i = \int_{A_i} \omega_2$, $b_i = \int_{B_i} \omega_2$, $T_{W'_\mu}\omega_2 = \sum_{i=1}^{k_\mu} b_i \sigma_{W'_\mu}(A_i) - a_i \sigma_{W'_\mu}(B_i)$, with reproducing differentials $\sigma_{W'_\mu}(A_i)$, $\sigma_{W'_\mu}(B_i)$ on $W'_\mu$ associated with cycles $A_i$ resp. $B_i$. By the theorem in [1]

$$(\omega_1, \omega_2^* - (T_{W'_\mu}\omega_2)^*)_{W'_\mu} = \int_{\partial W'_\mu} u(p)\bar{\omega}_2,$$

where $u(p)$ is a function defined separately on each component of $\partial W'_\mu$. If we put, (cf. (3.7)),

$$Q(\partial F_{3n_\mu}) = \sum_{i=1}^{m'(\mu)} \Gamma^i_\mu, \quad Q(\partial W_\mu) = \sum_{i=1}^{m'(\mu)} \sum_{j=1}^{N^i_\mu} \Gamma^{ij}_\mu, \quad Q(\partial W'_\mu) = \sum_{i=1}^{m'(\mu)} \sum_{j=1}^{N^i_\mu} \Gamma'^{ij}_\mu,$$

then, since $\omega_1, \omega_2 \in \Gamma_{\rm hse}(W)$, we have

$$\left| \int_{\partial W'_\mu} u(p)\bar{\omega}_2 \right| \leqslant \sum_{i,j} \left| \int_{\Gamma'^{ij}_\mu} u(p)\bar{\omega}_2 \right| \leqslant \sum_{i,j} \int_{\Gamma'^{ij}_\mu} |\omega_1| \int_{\Gamma'^{ij}_\mu} |\omega_2|.$$

Therefore by Lemma 4 we can obtain

$$|(\omega_1, \omega_2^* - (T_{W'_\mu}\omega_2)^*)| \leqslant \sum_i \sum_j \int_{\Gamma'^{ij}_\mu} |\omega_1| \int_{\Gamma'^{ij}_\mu} |\omega_2| \leqslant 2(1+h)(L(r_\mu) + M_\mu(\rho_\mu)) \to 0.$$

COROLLARY 1. *If we put $h=1$, Theorem 1 reduces to Theorem 1 in [4].*

LEMMA 5. *Under the conditions of Lemma 4, there exists an exhaustion $(W_\mu) \in \mathcal{E}_h$ such that the Riemann's bilinear relation holds for $\omega_1, \omega_2 \in \Gamma_{\rm hse}(W)$ where $\omega_2$ should satisfy the following conditions, (i) $\lim_{n\to\infty} \|T_n\omega_2\|_{W-\bar{F}_n} < \infty$, (ii) $\sum_{n=1}^{\infty} \|T_{n+1}\omega_2\|^2_{F_{n+2}-\bar{F}_n} < \infty$, where $a_i = \int_{A_i} \omega_2$, $b_i = \int_{B_i} \omega_2$, $T_n\omega_2 = \sum_{i=1}^{k_n} b_i \sigma(A_i) - a_i \sigma(B_i)$ with reproducing differentials $\sigma(A_i)$, $\sigma(B_i)$ on $W$ associated with cycles $A_i$, $B_i$ respectively, and $\{A_i, B_i\} = H.B.(F_n)$.*

PROOF. We put

$$\tilde{L}_n^i(r)=\Big(\int_{\Gamma_r^i}|\omega_1|+|\omega_2|+|T_n\omega_2|\Big)^2,\quad \tilde{L}_n(r)=\sum_{i=1}^{m(n)}\tilde{L}_n^i(r),$$

$$\tilde{L}(r)=\tilde{L}_n(r)\quad\text{for } \sum_j^{n-1}M_j\leqslant r<\sum_j^{n}M_j \text{ (cf. Lemma 2)},$$

$$\tilde{M}_\mu^i(\rho_\mu)=\sum_{j,\nu}\Big(\int_{S_{\mu\nu}^{ij}(\rho_\mu)}|\omega_1|+|\omega_2|+|T_n\omega_2|\Big)^2,\quad \sum_i\tilde{M}_\mu^i(\rho_n)=\tilde{M}_\mu(\rho_\mu).$$

Then by the same way as in Lemma 4, we can get for $\omega_1,\omega_2\in\Gamma_{\mathrm{hse}}(W)$ a sequence $\{\gamma_\mu|u=r_\mu\}$ tending to $\mathfrak{J}$ such that $\lim_{\mu\to\infty}\tilde{L}(r_\mu)=0$, and $\lim_{\mu\to\infty}\tilde{M}_\mu(\rho_\mu)=0$. Therefore

$$(\omega_1,\omega_2^*)=(\omega_1,\omega_2^*-(T_\mu\omega_2)^*)_{W_\mu}+(\omega_1,\omega_2^*-(T_\mu\omega_2)^*)_{W-W_\mu}+(\omega_1,(T_\mu\omega_2)^*).$$

But $|(\omega_1,\omega_2^*-(T_\mu\omega_2)^*)_{W-W_\mu}|\leqslant\|\omega_1\|_{W-W_\mu}(\|\omega_1\|_{W-W_\mu}+\|T_\mu\omega_2\|_{W-W_\mu})\to 0.$

$$|(\omega_1,\omega_2^*-(T_\mu\omega_2)^*)_{W_\mu}|=|(\omega_1,\omega_2^*-(T_\mu\omega_2)^*)_{W'_\mu}|\leqslant c'(\tilde{L}(r_\mu)+\tilde{M}_\mu(\rho_\mu))\to 0.$$

Consequently $(\omega_1,\omega_2^*)=\lim_{\mu\to\infty}(\omega_1,(T_\mu\omega_2)^*)$. Q.E.D.

THEOREM 2. *If $W$ satisfies the conditions of Lemma 4, the Riemann's bilinear relation holds for $\omega_1,\omega_2\in\Gamma_{\mathrm{hse}}(W)$, where the one has only a finite number of non vanishing periods.*

REMARK. This is analogous to Theorem 2 in [4], but we have much more freedom for the choice of the homology basis though our surfaces are more restricted than theirs.

REMARK. If $W\in 0_{\mathrm{HD}}$ and satisfies the conditions of Lemma 4, Theorem 2 reduces to Theorem 7 in [5].

## References


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