173. *Remarks on Generalized Rings of Quotients*

By Tomoharu AKIBA
Department of Education, Kobe University
(Comm. by Zyoiti SUETUNA, M.J.A., Dec. 12, 1964)

Introduction. Let A be an integral domain and let B be an overring of A contained in the quotient field of A. Then B is called a *generalized ring of quotients* of A if B is flat as an A-module. It has been shown that generalized rings of quotients have similar properties to those of ordinary rings of quotients (see [2] and [6]). In §1 of this paper, we first generalize the results to the case where A is not necessarily an integral domain. Some of the proofs are adaptions of those of [6], but, in order to make this paper self-contained, we repeat them again. In §2, we give a counter example to the following conjecture of Richman in [6].

Let A be an integral domain and let B be a generalized ring of quotients of A not equal to A. Then there exists an x/y in B which is not in A, such that $(x, y)A$ is invertible.

The author wish to express his heartful thanks to Prof. M. Nagata for kind advices and for valuable suggestions.

§1. First of all, we list some well-known properties of flatness as lemmas without proofs (cf. [1], [3], [4]). Rings will mean always commutative rings with units.

Lemma 1. *Let R and R' be rings such that R' is an R-module. Then R' is R-flat if and only if the following condition is satisfied:*

If (y_i) and (c_i) are finite subsets of R' and R respectively, such that $\sum_i c_i y_i = 0$, then there exist a finite subset (x_j) of R' and a finite subset (b_{ij}) of R for which we have $\sum_i b_{ij} c_i = 0$ for each j, and $y_i = \sum_j b_{ij} x_j$ for each i.

Lemma 2. *Let R and R' be as above and assume that R' is R-flat. Let $\mathfrak{a}_1, \cdots, \mathfrak{a}_r$ be ideals of R. Then we have $\left(\bigcap_i \mathfrak{a}_i\right)R' = \bigcap_i \mathfrak{a}_i R'$.*

Let A be a ring. In this section, we shall denote by B an overring of A contained in the total quotient ring of A.

Theorem 1. *The following three conditions are equivalent to each other:*

(1) *B is A-flat.*

(2) *For any element b of B, we have $(A:b)B = B$, where $(A:b)$ denotes the set of elements a of A such that $ab \in A$. (It is evident that $(A:b)$ is an ideal of A.)*

(3) *For every prime ideal* $\mathfrak{p}'$ *of* B, *the canonical homomorphism* $\varphi_{\mathfrak{p}'}$ *from* $A_{\mathfrak{p}'\cap A}$ *in* $B_{\mathfrak{p}'}$ *is bijective.* (Cf. Theorems 1 and 2 in [6].)

Proof. Equivalence between (1) and (2): Assume that B is A-flat. Let $b=x/y$ ($x, y\in A$; y is not a zero divisor) be an element of B. Then $y(x/y)-x.1=0$. By Lemma 1, there exist a finite subset (b_j) of B and a finite subset (a_{1j}, a_{2j}) of A such that (1′) $a_{1j}y-a_{2j}x=0$ for each j, (2′) $x/y=\sum_j a_{1j}b_j$, and (3′) $1=\sum_j a_{2j}b_j$. From (1′), it follows that a_{2j} is in $(A:b)$ for every j, and (3′) asserts that $(A:b)B=B$. Thus (1) implies (2).*) Assume, conversely, that (2) is valid. Let (y_i) and (c_i) be finite subsets of B and A respectively, such that $\sum_i y_i c_i=0$. Since $(A:y_i)B=B$ for each i by the condition (2), we have $\left(\bigcap_i (A:y_i)\right)B\supseteqq\prod_i (A:y_i)B\supseteqq B$, and $\left(\bigcap_i (A:y_i)\right)B=B$. So there are finite subsets (a_j) and (x_j) of $\bigcap_i (A:y_i)$ and B respectively, for which we have $\sum_j a_j x_j=1$. Then $b_{ij}=y_j a_j$ is in A for each i and j, $y_i=\sum_j y_i a_j x_j=\sum_j b_{ij}x_j$ for every i, and $0=\sum_i c_i y_i a_j=\sum_i c_i b_{ij}$ for each j. By Lemma 1, this shows that B is A-flat. Hence (2) implies (1).

Equivalence between (2) and (3): assume that (2) is true. Let $\mathfrak{p}'$ be an arbitrary prime ideal of B and set $\mathfrak{p}=\mathfrak{p}'\cap A$. If $\varphi_{\mathfrak{p}'}(a/s)=0$ for an a/s of $A_{\mathfrak{p}}(a\in A, s\in A-\mathfrak{p})$, then we have $as'=0$ with an s' in $B-\mathfrak{p}'$. By the condition (2), $(A:s')B=B$ and then $(A:s')\nsubseteq\mathfrak{p}$. Hence there is an element t in $(A:s')$ such that $t\notin\mathfrak{p}$. Then $as't=0$ with $s't\in A-\mathfrak{p}$, which shows that $a/s=0$ in $A_{\mathfrak{p}}$, whence $\varphi_{\mathfrak{p}'}$ is injective. Next, let b/s' be an arbitrary element of $B_{\mathfrak{p}'}(b\in B, s'\in B-\mathfrak{p}')$. Since $(A:b)B=B$ and $(A:s')B=B$, we have $((A:b)\cap(A:s'))B=B$ and so $(A:b)\cap(A:s')\nsubseteq\mathfrak{p}$. Then there is an element s in $(A:b)\cap(A:s')$ which is not in $\mathfrak{p}$. For the s, we have $ss'\in A-\mathfrak{p}$ and $bs\in A$, so bs/ss' may be considered as an element of $A_{\mathfrak{p}}$. It is obvious that $\varphi_{\mathfrak{p}'}(bs/ss')=b/s'$, which shows that $\varphi_{\mathfrak{p}'}$ is surjective. Thus (2) implies (3). Conversely, assume that (3) is satisfied, and suppose that there is a $b\in B$ such that $(A:b)B\neq B$. Then there is a prime ideal $\mathfrak{p}'$ of B containing $(A:b)B$, and so we have $\mathfrak{p}\supseteqq(A:b)$ for $\mathfrak{p}=\mathfrak{p}'\cap A$. Since $\varphi_{\mathfrak{p}'}$ is surjective by the condition (3), we can take $a\in A$ and $s\in A-\mathfrak{p}$ so that $\varphi_{\mathfrak{p}'}(a/s)=b/1$, which implies that $(a-bs)s'=0$ for some $s'\in B-\mathfrak{p}'$. From the assumption that B is contained in the total quotient ring of A, it follows that there is a $t\in(A:b)$ which is not a zero divisor in A and so in B. Then $(at-bst)s'=0$, which shows that $\varphi_{\mathfrak{p}'}(at/1)=\varphi_{\mathfrak{p}'}(bst/1)$. So there is an $r\in A-\mathfrak{p}$ such that $(at-bst)r=0$ because $\varphi_{\mathfrak{p}'}$ is injective by (3). From this we have $ar-bsr=0$, since t is not a zero divisor. Hence we have $sr\in A-\mathfrak{p}$ and $sr\in(A:b)$, which is a contradiction because $\mathfrak{p}\supseteqq(A:b)$. Thus we have $(A:b)B=B$ for all

*) This part of the proof is the same as that of [6].

$\mathfrak{b}$ of B, and (3) implies (2).

Adapting [6], an overring B of a ring A contained in the total quotient ring of A is said to be a *generalized ring of quotients* of A if B is A-flat.

Corollary 1. *Let B be a generalized ring of quotients of A. Then for any overring C of A contained in B, B is a generalized ring of quotients of C.* (Cf. Lemma 2 in [6].)

The proof is straightforward and we omit it.

Corollary 2. *If a generalized ring of quotients B of A is integral over A, then $A=B$.* (Cf. Proposition 2 in [6].)

The proof follows directly from Theorem 1 and the fact that if B is integral over A, then the extended ideal of a proper ideal of A to B is again proper.

Corollary 3. *Let B be a generalized ring of quotients of A, and let A^* and B^* be integral closures of A and B respectively, in the total quotient ring of A. Then $B^*=B[A^*]$, and B^* is a generalized ring of quotients of A^*. In particular, if A is integrally closed in its total quotient ring, then B is also integrally closed in its total quotient ring.* (Cf. Proposition 1 and Corollary in [6].)

Proof. Let b^* be an element of B^*, then $b^{*n}+b_1b^{*n-1}+\cdots+b_n=0$ with $b_i\in B$. From Theorem 1, it follows that $(A:b_i)B=B$ for every i and so $\left(\bigcap_i (A:b_i)\right)B=B$. Then there are finite subsets (a_j) and (c_j) of $\bigcap_i (A:b_i)$ and B respectively, such that $\sum_j a_jc_j=1$. Since a_jb^* is in A^* for every j, we have $b^*=\sum_j a_jc_jb^*\in B[A^*]$, which shows that $B^*\subseteqq B[A^*]$. The converse inclusion being obvious, $B^*=B[A^*]$. Since, under the above notations, $(A^*:b^*)\supseteqq\bigcap_i (A:b_i)$, we see that B^* is a generalized ring of quotients of A^* by Theorem 1 and the definition. The last assertion is trivial.

Theorem 2. *Let B be a generalized ring of quotients of A. Then:*

(1) *For any ideal $\mathfrak{b}$ of B, we have $(\mathfrak{b}\cap A)B=\mathfrak{b}$. In particular, prime ideals of B are generated by prime ideals of A.*

(2) *Let $\mathfrak{q}$ be a primary ideal of A belonging to a prime ideal $\mathfrak{p}$ and such that $\mathfrak{q}B\neq B$. Then $\mathfrak{p}B\neq B$, $\mathfrak{p}B$ is a prime ideal, $\mathfrak{q}B$ is primary to $\mathfrak{p}B$, $\mathfrak{p}B\cap A=\mathfrak{p}$, and $\mathfrak{q}B\cap A=\mathfrak{q}$.* (Cf. Theorem 3 in [6].)

Proof. Let b be an element of $\mathfrak{b}$. Since $(A:b)B=B$ by Theorem 1, there are finite subsets (a_i) and (b_i) of $(A:b)$ and B respectively, such that $\sum_i a_ib_i=1$. Then $a_ib\in\mathfrak{b}\cap A$ for every i, and $b=\sum_i a_ib_ib$ is in $(A:b)B$, which shows that $(\mathfrak{b}\cap A)B\supseteqq\mathfrak{b}$. Since the converse inclusion is clear, we have $(\mathfrak{b}\cap A)B=\mathfrak{b}$. Thus (1) is proved. The first assertion in (2) is trivial. Next, we shall prove the other assertions in (2). If $q \in \mathfrak{q}B \cap A$, then $q = \sum_i q_i b_i$ with $q_i \in \mathfrak{q}$, $b_i \in B$. Since $(A : b_i)B = B$ for each i by Theorem 1, we have $\left(\bigcap_i (A : b_i)\right)B = B$. From $\mathfrak{p}B \neq B$, it follows that $\mathfrak{p} \not\supseteq \bigcap_i (A : b_i)$, hence there is an a in $\bigcap_i (A : b_i)$ which is not in $\mathfrak{p}$. Then $aq \in \mathfrak{q}$ and, since $a \notin \mathfrak{p}$, we have $q \in \mathfrak{q}$. This shows that $\mathfrak{q}B \cap A \subseteq \mathfrak{q}$. On the other hand, that $\mathfrak{q}B \cap A \supseteq \mathfrak{q}$ is clear and we have $\mathfrak{q}B \cap A = \mathfrak{q}$. As a particular case where $\mathfrak{q} = \mathfrak{p}$, we have $\mathfrak{p}B \cap A = \mathfrak{p}$. Now, let b and b' be elements of B such that $bb' \in \mathfrak{q}B$ and $b' \notin \mathfrak{q}B$. Then there is an $a' \in (A : b')$ such that $a'b' \notin \mathfrak{q}$. In fact, otherwise, we would have $b' \in b'(A : b')B \subseteq \mathfrak{q}B$ because $(A : b')B = B$, which is a contradiction. Furthermore, since $(A : b)B = B$ there are $a_1, \cdots, a_r$ in $(A : b)$ such that $\mathfrak{a}B = B$ where $\mathfrak{a} = (a_1, \cdots, a_r)A$. Then it is obvious that for any positive integer n, $\mathfrak{a}^n B = B$. On the other hand, we have $a_i a' b b' = a_i b a' b' \in \mathfrak{q}B \cap A = \mathfrak{q}$ for $i = 1, \cdots, r$. Since $a'b' \notin \mathfrak{q}$ and since $\mathfrak{q}$ is a primary ideal, it follows that there is a positive integer n_i such that $(a_i b)^{n_i} \in \mathfrak{q} (i = 1, \cdots, r)$. Then, taking a positive integer n to be $n \geqq \max\{rn_i\}$, we have $b^n \in b^n \mathfrak{a}^n B \subseteq \mathfrak{q}B$ as can be easily seen, which shows that $\mathfrak{q}B$ is a primary ideal. Applying this to the case where $\mathfrak{q} = \mathfrak{p}$, we see that $\mathfrak{p}B$ is a prime ideal because in that case n can be taken to be 1. Any element of $\mathfrak{p}$ being nilpotent modulo $\mathfrak{q}$, elements of $\mathfrak{p}B$ are also nilpotent modulo $\mathfrak{q}B$, whence $\mathfrak{q}B$ belongs to $\mathfrak{p}B$. Thus the proof of Theorem 2 is complete.

Corollary 1. *If A is Noetherian, then any generalized ring of quotients of A is Noetherian.* (Cf. Corollary of Theorem 3 in [6].)

This follows immediately from the above theorem and the well-known theorem of Cohen (see (3.4) of Chap. 1 in [4]).

The following corollary is an immediate consequence of Theorem 2 and Lemma 2.

Corollary 2. *Let B be a generalized ring of quotients of A and let $\mathfrak{q}_1, \cdots, \mathfrak{q}_r$ be primary ideals of A such that $\mathfrak{q}_i B \neq B$ for every i. Set $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$. Then $\mathfrak{a}B = \mathfrak{q}_1 B \cap \cdots \cap \mathfrak{q}_r B$ and $\mathfrak{a}B \cap A = \mathfrak{a}$.* (Cf. Theorem 3 in [6].)

§ 2. We shall give a counter example to the conjecture of Richman (see Introduction) in the case where A is a local integral domain. In that case, condition that $(x, y)A$ is invertible implies that $(x, y)A$ is principal, say $(x, y)A = zA$. Then $(x/z, y/z)A = A$. Since A is local, one of x/z and y/z is a unit, whence $(x, y)A = xA$ or $(x, y)A = yA$. But x/y is not in A by our assumption, so we have $(x, y)A = xA$. Then y/x is in A and is invertible in B.

Therefore, for our purpose, it is enough to construct a local integral domain A and a generalized ring of quotients B of A such that $B \neq A$ and no non-unit of A is invertible in B. In the following, the notations will be as in [5].

Let C be a non-singular plane cubic curve defined over a field k_0 and let P be a generic point of C over k_0, and let k be a field containing $k_0(P)$. Then the homogeneous coordinate ring $R_0=k[x, y, z]$ of C over k is normal. Let $R=k[x, y, z]_{(x,y,z)}$ and $R'=\bigcup_n \mathfrak{p}^{-n}$ ($\mathfrak{p}$-transform of R in the sense of [5]), where $\mathfrak{p}$ is the homogeneous prime ideal of R corresponding to P.

We shall show that $A=R$ and $B=R'$ give the required example.*)

First, we shall prove that no non-unit of R is invertible in R'. Suppose that there is an f of R such that f is non-unit in R and $f^{-1}\in R'$. Then $\mathfrak{p}^n \subseteqq fR$ for some n and the normality of R implies that fR is primary to $\mathfrak{p}$. Therefore $fR=\mathfrak{p}^{(m)}$ (m-th symbolic power of $\mathfrak{p}$) for a suitable m. Since $\mathfrak{p}^{(m)}=\mathfrak{p}_0^{(m)}R$ ($\mathfrak{p}_0=\mathfrak{p}\cap R_0$) and since $\mathfrak{p}_0^{(m)}$ is homogeneous, we may assume that f is a homogeneous element of R_0. Then $fR_0=\mathfrak{p}_0^{(m)}$, and this shows that the intersection of the hypersurface $f=0$ with C is mP, which is a contradiction because P is a generic point and C is of positive genus.

Next, we shall prove that R' is R-flat.

Lemma. *Let $\mathfrak{O}$ be an integral domain and let $\mathfrak{a}$ be an ideal of $\mathfrak{O}$. Set $\mathfrak{O}'=\bigcup_n \mathfrak{a}^{-n}$. Then there exists a one to one correspondence between prime ideals $\mathfrak{q}'$ of $\mathfrak{O}'$ and prime ideals $\mathfrak{q}$ of $\mathfrak{O}$ except those containing $\mathfrak{a}\mathfrak{O}'$ and $\mathfrak{a}$ respectively, in such a way that $\mathfrak{q}'$ corresponds to $\mathfrak{q}=\mathfrak{q}'\cap\mathfrak{O}$. In the case we have $\mathfrak{O}'_{\mathfrak{q}'}=\mathfrak{O}_{\mathfrak{q}}$.* (Cf. Lemma 3 of § 1 in [5].)

By the above lemma and Theorem 1, it is sufficient to prove that $\mathfrak{p}R'=R'$. Let $R'_0=\bigcup_n \mathfrak{p}_0^{-n}$. Since P is rational over k, $\mathfrak{p}_0$ can be generated by linear forms. If t is a linear form contained in $\mathfrak{p}_0$, then any element of R'_0 is of the form q/t^n with $q\in t^nR_0:\mathfrak{p}_0^n$ for a suitable n (see [5]). Assume that q is homogeneous of degree d. For a homogeneous element g of R_0, we denote by D_g the divisor of C which is cut out by the hypersurface $g=0$. Then $q\in t^nR_0:\mathfrak{p}_0^n$ if and only if $D_q+nP\succ nD_t$. Since C is a non-singular plane curve, the system of hypersurface sections of a given degree is complete. Furthermore, since the genus of C is 1, for any divisor D of degree greater than 1, the complete system $|D|$ has no fixed points. Then it follows that, taking n and d so that $3d-2n>2$ and $n>d$, we can find two homogeneous forms q_1 and q_2 of degree d in $t^nR_0:\mathfrak{p}_0^n$ such that $D_{q_1}-nD_t$ and $D_{q_2}-nD_t$ have no common points except P. Taking two linear forms h_1 and h_2 belonging to $\mathfrak{p}_0$ so that $D_{q_1h_1}-nD_t$ and $D_{q_2h_2}-nD_t$ have no common points except P, we set $a_1=q_1h_1^{n-d}/t^n$ and $a_2=q_2h_2^{n-d}/t^n$.

On the other hand, $C-\{P\}$ is an affine curve and so we denote its affine ring by R^*. Then a_1 and a_2 are contained in R^* and

*) This example was obtained following a suggestion made by Prof. Nagata

$a_1R^* + a_2R^* = R^*$ because a_1 and a_2 have no common zeros. Obviously, $R' \supseteqq R^*$ and, since $q_i/t^n \in R'$ and $h_i \in \mathfrak{p}$, we have $a_i \in \mathfrak{p}R'$ $(i=1, 2)$. Then the relation $a_1R^* + a_2R^* = R^*$ implies that $\mathfrak{p}R' = R'$, as we wanted.

References

[1] N. Bourbaki: Algèbre Commutative. Hermann, Paris (1961).
[2] L. Budach: Über eine Charakterisierung der Grellschen Schemata. Math. Nachr., **27**, 5-6 (1964).
[3] M. D. Lazard: Sur les modules plats. C. R. Acad. Sci. Paris, **258** (1964).
[4] M. Nagata: Local Rings. Interscience, New York (1962).
[5] ——: A treatise on 14th problem of Hilbert. Mem. Coll. Sci.U niv. Kyoto, **30** (1) (1956); Corrections, ibid, **30** (3) (1957).
[6] F. Richman: Generalized quotient rings, forthcoming.
[7] R. Walker: Algebraic Curves. Princeton University Press (1950).