167. *Another Proof of a Theorem Concerning the Greatest Semilattice-Decomposition of a Semigroup*

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1. **Introduction.** For any semigroup S, consider any congruence ρ on S such that S/ρ is a semilattice, i.e., a commutative idempotent semigroup. Such a ρ is called a semilattice-congruence or simply s-congruence. As is well known, there is the smallest s-congruence ρ_0 on S in the sense of inclusion [1–7]. Let $L=S/\rho_0$ and let S_α, $\alpha\in L$, be a congruence class modulo ρ_0:

$$S=\bigcup_{\alpha\in L} S_\alpha,\quad S_\alpha\cap S_\beta=\Box,\quad \alpha\neq\beta.$$

If the cardinal number $|L|$ of L is exactly 1, that is, ρ_0 is the universal relation on S, then S is called s-indecomposable; if $|L|>1$, then S is s-decomposable. The partition of S due to ρ_0 is called the greatest s-decomposition of S, and S/ρ_0 is called the greatest s-homomorphic image of S.

Theorem. *In the greatest s-decomposition of a semigroup S, each congruence class S_α is s-indecomposable.*

This theorem was proved by the author [4] and recently stated by Petrich in [2] without proof. The purpose of this paper is to give a proof of this theorem from somewhat different point of view. Proposition 1 below can be proved by using the above theorem, but here we are going to prove Proposition 1 directly and then to prove the above theorem by using it.

2. **Preliminaries.** Let $a_1,\cdots,a_n$ be elements of a semigroup S. If an element a of S is the product of all of $a_1,\cdots,a_n$ admitting repeated use, then a is said to be fully generated by $a_1,\cdots,a_n$. The set G of all the elements of S which are fully generated by $a_1,\cdots,a_n$ is a non-empty subsemigroup of S. G is called the subsemigroup of S fully generated by $a_1,\cdots,a_n$.

Let $\mathfrak{F}_0$ be the free semigroup generated by n distinct letters $a_1,\cdots,a_n$ in the usual sense, and $\mathfrak{F}$ be the subsemigroup of $\mathfrak{F}_0$ fully generated by $a_1,\cdots,a_n$. $\mathfrak{F}$ is composed of all words any one of which contains all of $a_1,\cdots,a_n$.

Let ρ be any s-congruence on $\mathfrak{F}$. We denote by φ the natural mapping of $\mathfrak{F}$ upon $\mathfrak{F}/\rho$, that is, for $A\in\mathfrak{F}, A\varphi$ of $\mathfrak{F}/\rho$ is the congruence class modulo ρ containing A. For convenience of the proof, we define the partial ordering $\leqq$ in the semilattice $\mathcal{F}/\rho$ in the usual way:

$$A\varphi \geqq B\varphi \text{ iff } (B\varphi)(C\varphi)=A\varphi \text{ for some } C\in\mathcal{F}$$

then $A\rho B$ iff $A\varphi=B\varphi$.

The letters x_i in Lemmas 1 and 3 below denote some of $a_1, \cdots, a_n$; and x_i and x_j may happen to be the same.

Lemma 1. *Let $x_1\cdots x_i\cdots x_m$ be a word in $\mathcal{F}$. Then*

$$x_1\cdots x_{i-1}x_i x_{i+1}\cdots x_m \,\rho\, x_i x_{i+1}\cdots x_m x_1\cdots x_{i-1},\quad 1\leqq i\leqq m.$$

Proof. Let $X=x_1\cdots x_{i-1}x_ix_{i+1}\cdots x_m$, $Y=x_ix_{i+1}\cdots x_mx_1\cdots x_{i-1}$, and let $Z=x_1\cdots x_{i-1}$, $U=x_i\cdots x_m$. Then ZUZ, $UZU\in\mathcal{F}$ and we have

$$X\rho X^3=(ZUZ)(UZU)\rho(UZU)(ZUZ) \text{ by commutativity}$$
$$=Y^3\rho Y. \qquad \text{q.e.d.}$$

Let $A\in\mathcal{F}$ and let x be any one of the letters $a_1, \cdots, a_n$. We define words Ax and xA in $\mathcal{F}$ as follows:

If $A=x_1\cdots x_m$, then $Ax=x_1\cdots x_mx$, $xA=xx_1\cdots x_m$. More generally, we can define words AZ and ZA in $\mathcal{F}$ for $Z\in\mathcal{F}_0$. Let $Z=z_1\cdots z_k$ where $z_1, \cdots, z_k$ are some of $a_1, \cdots, a_n$.

$$AZ=x_1\cdots x_mz_1\cdots z_k, \qquad ZA=z_1\cdots z_kx_1\cdots x_m.$$

Clearly $A(ZU)=(AZ)U$, $(ZU)A=Z(UA)$, $Z, U\in\mathcal{F}_0$. According to Lemma 1, we see $Ax\rho xA$ and $AZ\rho ZA$, for $Z\in\mathcal{F}_0$.

Lemma 2. *Let $A, B\in\mathcal{F}$, and x be any one of $a_1, \cdots, a_n$. Then $A\rho B$ implies $Ax\rho Bx$ and $xA\rho xB$.*

Proof. With using $A\varphi\geqq B\varphi$ and its compatibility, we have

$$(Ax)\varphi=((Ax)\varphi)^2=(AxAx)\varphi=(A\varphi)((xAx)\varphi)\geqq(B\varphi)((xAx)\varphi)$$
$$=(BxAx)\varphi=(Bx)\varphi((Ax)\varphi)\geqq(Bx)\varphi.$$

In the same way, we have $(Bx)\varphi\geqq(Ax)\varphi$. The dual case is obtained immediately by the above remark.

Lemma 3. *Again let x be any one of $a_1, \cdots, a_n$. Let $A\in\mathcal{F}$. Then $Ax\rho A$ and $xA\rho A$.*

Proof. Since $A\varphi=A^2\varphi$, by Lemma 2,

$$(Ax)\varphi=(AAx)\varphi=(A\varphi)((Ax)\varphi)\geqq A\varphi.$$

By Lemma 1, we can find $B\in\mathcal{F}_0$ such that $A\varphi=(xB)\varphi$.

$$A\varphi=A^3\varphi=(A\varphi)(A\varphi)(A\varphi)=(A\varphi)((xB)\varphi)(A\varphi)$$
$$=((Ax)\varphi)((BA)\varphi)\geqq(Ax)\varphi.$$

Thus we have $(Ax)\varphi=A\varphi$, or $Ax\rho A$.

3. Propositions and Theorems.

Proposition 1. *$\mathcal{F}$ is s-indecomposable.*

Proof. Recalling that ρ is any s-congruence on $\mathcal{F}$. Let A and B be any elements of $\mathcal{F}$. According to Lemma 3,

$$Ax\rho A \text{ if } x \text{ is any one of } a_1, \cdots, a_n.$$

Immediately we have

$$AB\rho A \text{ for all } A, B\in\mathcal{F} \text{ and hence } A\rho AB\rho BA\rho B.$$

Thus the proposition has been proved.

As the application of Proposition 1, we have

Proposition 2. *Let S be any semigroup, $a_1, \cdots, a_n$ be elements of S, and G be the subsemigroup of S fully generated by $a_1, \cdots, a_n$. Then G is s-indecomposable.*

Proof. Let G' be an s-homomorphic image of G, and $\mathcal{F}$ be the semigroup fully generated by the letters $a_1, \cdots, a_n$, which is discussed in the preceding section. Since $\mathcal{F}$ is homomorphic onto G, G' is a homomorphic image of the greatest s-homomorphic image of $\mathcal{F}$. On the other hand, $\mathcal{F}$ is s-indecomposable by Proposition 1 and hence $|G'|=1$. Thus G is s-indecomposable.

Proof of Theorem. Let ρ_0 be the smallest s-congruence on S so that $S=\bigcup_{\alpha\in S/\rho_0} S_\alpha$. Let σ be the smallest s-congruence on S_α. We are to prove that σ is universal on S_α. Suppose $a\rho_0 b$ and $a, b\in S_\alpha$. Then there are a finite number of elements

$$a=b_1, b_2, \cdots, b_{m-1}, b_m=b$$

such that b_i and b_{i+1} are related in such a fashion that one of the following cases (1), (2), and (3) happens [5, 6, 7].

$$\text{Case I}\quad \begin{cases} b_i=zxyu \\ b_{i+1}=zyxu \end{cases}\quad \text{for some } z, u\in S^1;\ x, y\in S$$

$$\text{Case II}\quad \begin{cases} b_i=zx^2u \\ b_{i+1}=zxu \end{cases}\quad \text{for some } z, u\in S^1;\ x\in S$$

$$\text{Case III}\quad \begin{cases} b_i=zxu \\ b_{i+1}=zx^2u \end{cases}\quad \text{for some } z, u\in S^1;\ x\in S$$

where $S^1=S\cup\{1\}$, 1 being the adjoined two-sided identity and where x, y, z, u depend on i. Now G_i is defined as follows: G_i is the subsemigroup of S fully generated by

x, y, z, u　　in the Case I

x, u, z　　in the Case II and III

where if z or u is 1, it is omitted. By the definition of ρ_0, any two elements of G_i are ρ_0-related, that is,

$$G_i\subseteq S_\alpha \text{ if } b_i, b_{i+1}\in S_\alpha.$$

Obviously the restriction of σ to G_i is an s-congruence on G_i. In the consequence of Proposition 2, σ becomes universal on G_i. Therefore $b_i\sigma b_{i+1}$ $(i=1, \cdots, m-1)$ in S_α. By transitivity we get $a\sigma b$. This completes the proof of the theorem.

4. Remark. We can prove Proposition 1 in another way: With using Lemmas 1 and 2, we prove

Lemma 4. *Let $A=x_1\cdots x_i\cdots x_m\in\mathcal{F}$. Then*

$$Ax_i\rho x_1\cdots x_i^2\cdots x_m\rho x_iA.$$

Let W be a word in $\mathcal{F}$. The sum of all the exponents of a letter x in W is called the total exponent of x in W. For example, if $W=x^2yz^2xz^2$, the total exponent of x in W is 3. Fixing the order of the letters $a_1, \cdots, a_n$, the total exponent of a_i in W is denoted by t_i. If $A \in \mathcal{F}$, all t_i are positive integers. We define $\tau(A)$ as follows:

$$\tau(A)=(t_1, \cdots, t_n), \quad A \in \mathcal{F}.$$

Also if $\tau(B)=(t'_1, \cdots, t'_n)$, then $\tau(A)=\tau(B)$ iff $t_i=t'_i$ $(i=1, \cdots, n)$. If $A \in \mathcal{F}_0$, some t_i may happen to be 0.

From Lemma 4, immediately

Lemma 5. *Let $A \in \mathcal{F}$; $Z, U \in \mathcal{F}_0$. If $\tau(Z)=\tau(U)$, then $AZ\rho AU$ and $ZA\rho UA$.*

Now, define a congruence ξ on $\mathcal{F}$ as follows:

$$A\xi B \text{ iff } \tau(A)=\tau(B).$$

$\tau(\mathcal{F})$ is the direct sum of the semigroups I of all positive integers with usual vector addition:

$$\tau(\mathcal{F})=\underbrace{I\oplus I\oplus \cdots \oplus I}_{n}.$$

Clearly τ is a homomorphism of $\mathcal{F}$ onto $\tau(\mathcal{F})$, and $\tau(\mathcal{F})$ is a commutative archimedean semigroup and hence idempotent-indecomposable. Suppose $A, B \in \mathcal{F}$ and $A\xi B$. Then by Lemma 5,

$$A\rho AA\rho AB\rho BA\rho BB\rho B.$$

Thus we have proved $\xi \subseteq \rho$, from which we can conclude that ρ is universal on $\mathcal{F}$.

Also we add that the subsemigroup G of S fully generated by a finite number of elements of S is closely related to the concept of P-subsemigroup due to Yamada [6, 7], and to that of N-classes due to Petrich [2].

References

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