9. Two Tauberian Theorems for (J, p_n) Summability

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§1. The present note is a continuation of a previous paper by the author [4]. We suppose throughout that

$$p_n \geq 0, \qquad \sum_{n=0}^{\infty} p_n = \infty,$$

and that the radius of convergence of the power series

$$p(x) = \sum_{n=0}^{\infty} p_n x^n$$

is 1. Given any series

$$(1) \qquad \sum_{n=0}^{\infty} a_n,$$

with the sequence of partial sums $\{s_n\}$, we shall use the notation:

$$(2) \qquad p_s(x) = \sum_{n=0}^{\infty} p_n s_n x^n.$$

If the series (2) is convergent in the open interval (0, 1), and if

$$\lim_{x \to 1-0} \frac{p_s(x)}{p(x)} = s,$$

we say that the series $\sum_{n=0}^{\infty} a_n$ or the sequence $\{s_n\}$ is summable (J, p_n) to s. As is well known, this method of summability is regular. (See, Borwein [1], Hardy [2], p. 80.) We shall prove, in this note, the following

Theorem 1. *Suppose that*

$$(3) \qquad p_n = O\left(\frac{1}{n}\right)$$

with $p_n > 0$. *Suppose that the series* (1) *is summable* (J, p_n) *to* s, *and that*

$$(4) \qquad a_n = o\left(\frac{p_n}{P_n}\right),$$

where

$$P_n = p_0 + p_1 + \cdots + p_n, \qquad n = 0, 1, \cdots.$$

Then (1) *converges to* s.

Proof. From (3) and (4) we can choose m such that, for $n > m$,

$$(5) \qquad np_n \leq M^{1)}$$

and

1) We use M to denote a constant, possibly different at each occurrence.

$$|a_n| \leq \varepsilon \frac{p_n}{P_n} \tag{6}$$

simultaneously, where ε is a positive number as small as we please.

First we shall prove the condition (3) implies

$$\frac{\sum_{n=0}^{m} p_n}{\sum_{n=0}^{\infty} p_n\left(1-\frac{1}{m}\right)^n} = O(1) \qquad \text{for } m \to \infty. \tag{7}$$

From (5) we have

$$\begin{aligned}\sum_{n=m+1}^{\infty} p_n\left(1-\frac{1}{m}\right)^n &\leq M \sum_{n=m+1}^{\infty} \frac{1}{n}\left(1-\frac{1}{m}\right)^n \\ &\leq \frac{M}{m} \sum_{n=0}^{\infty}\left(1-\frac{1}{m}\right)^n \\ &= M,\end{aligned}$$

hence

$$\begin{aligned}&\left|\sum_{n=0}^{m} p_n - \sum_{n=0}^{\infty} p_n\left(1-\frac{1}{m}\right)^n\right| \\ &\qquad \leq M + (p_1 + 2p_2 + \cdots + mp_m)/m,\end{aligned}$$

since, for $0<x<1$,

$$0 < p_n(1-x^n) < (1-x)np_n.$$

Since we assume

$$np_n = O(1),$$

we get

$$\sum_{n=1}^{m} np_n = O(m),$$

(see, e.g., Hobson [3] p. 7). Therefore we obtain

$$\left|\sum_{n=0}^{m} p_n - \sum_{n=0}^{\infty} p_n\left(1-\frac{1}{m}\right)^n\right| \leq M,$$

provided m be chosen sufficiently large. From this estimation we get easily

$$\lim_{m\to\infty} \frac{\sum_{n=0}^{m} p_n}{\sum_{n=0}^{\infty} p_n\left(1-\frac{1}{m}\right)^n} = 1,$$

and also (7) *a fortiori*.

Now we have, for $0<x<1$,

$$\begin{aligned}s_m - \frac{p_s(x)}{p(x)} &= \frac{\sum_{n=0}^{m-1}(s_m - s_n)p_n x^n}{\sum_{n=0}^{\infty} p_n x^n} + \frac{\sum_{n=m+1}^{\infty}(s_m - s_n)p_n x^n}{\sum_{n=0}^{\infty} p_n x^n} \\ &= I + J, \quad \text{say.}\end{aligned}$$

If x be chosen to be equal to $1-\frac{1}{m}$, we obtain

$$(8) \qquad I=o(1) \quad \text{for} \quad m\to\infty,$$

from (4) and (7) (see Ishiguro [4]).

Next we shall estimate J. From (6) and (5) we have

$$\begin{aligned} |s_m-s_n| &\leq \varepsilon\left\{\frac{p_{m+1}}{P_{m+1}}+\frac{p_{m+2}}{P_{m+2}}+\cdots+\frac{p_n}{P_n}\right\} \\ &\leq \frac{\varepsilon}{P_m}\{p_{m+1}+p_{m+2}+\cdots+p_n\} \\ &\leq \frac{\varepsilon M}{P_m}\left\{\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{n}\right\} \\ &\leq \frac{\varepsilon M}{P_m}\cdot\frac{n}{m}, \end{aligned}$$

hence

$$\begin{aligned} |J| &\leq \frac{\frac{\varepsilon M}{mP_m}\sum_{n=m+1}^{\infty} np_n x^n}{\sum_{n=0}^{\infty} p_n x^n} \\ &\leq \frac{\varepsilon P_m \frac{M}{mP_m^2}\sum_{n=m+1}^{\infty}\left(1-\frac{1}{m}\right)^n}{\sum_{n=0}^{\infty} p_n\left(1-\frac{1}{m}\right)^n} \end{aligned}$$

when $x=1-\frac{1}{m}$. Thus we get, from (7),

$$(9) \qquad \begin{aligned} |J| &\leq \varepsilon\frac{M}{mP_m^2}\sum_{n=0}^{\infty}\left(1-\frac{1}{m}\right)^n \\ &\leq \varepsilon \end{aligned}$$

for sufficiently large m.

Letting m increase indefinitely, we have

$$\lim_{m\to\infty} s_m = \lim_{x\to 1-0}\frac{p_s(x)}{p(x)}=s$$

from (8) and (9), which proves the theorem.

As in the previous paper [4], we obtain the following

Corollary. *Suppose that there exist two numbers* σ, M *such that*

$$0<\frac{\sigma}{n+1}\leq p_n\leq\frac{M}{n+1}, \qquad n=0,1,\cdots.$$

Suppose that the series (1) *is summable* (J, p_n) *to* s, *and that*

$$a_n=o\left(\frac{1}{n\log n}\right).$$

Then (1) *converges to* s.

§2. We shall prove here the following

Theorem 2. *Suppose that*

$$\frac{\sum_{n=0}^{m} p_n}{\sum_{n=0}^{\infty} p_n\left(1-\frac{1}{m}\right)^n}=O(1) \qquad \text{for } m\to\infty, \tag{7}$$

and that

$$\{p_n\} \text{ decreases monotonically} \tag{10}$$

with $p_n>0$. *Suppose that the series* (1) *is summable* (J, p_n) *to* s, *and that*

$$a_n=o\left(\frac{p_n}{P_n}\right). \tag{4}$$

Then (1) *converges to* s.

Proof. If $\{p_n\}\searrow\sigma$, $\sigma>0$, this theorem is a special case of Corollary of Theorem 3 in the previous paper [4], hence the condition (7) is unnecessary.

As in the proof of Theorem 1, we put

$$s_m-\frac{p_s(x)}{p(x)}=I+J,$$

then we have, from (4) and (7),

$$I=o(1) \qquad \text{for } m\to\infty,$$

when $x=1-\frac{1}{m}$.

Now we have, from (6),

$$|s_m-s_n|\le\varepsilon\left\{\frac{p_{m+1}}{P_{m+1}}+\frac{p_{m+2}}{P_{m+2}}+\cdots+\frac{p_n}{P_n}\right\}$$
$$\le\varepsilon\frac{P_n}{P_m},$$

hence

$$|J|\le\frac{\varepsilon\frac{1}{P_m}\sum_{n=m+1}^{\infty}P_n p_n x^n}{\sum_{n=0}^{\infty}p_n x^n}$$
$$\le\frac{\varepsilon\frac{p_m}{P_m}\sum_{n=m+1}^{\infty}P_n x^n}{\sum_{n=0}^{\infty}p_n x^n}$$

from (10). As in the proof of Theorem 1, we have, from (7),

$$|J|\le\varepsilon M\frac{p_m}{P_m^2}\sum_{n=m+1}^{\infty}P_n\left(1-\frac{1}{m}\right)^n$$

when $x=1-\frac{1}{m}$. Here we put

$$R_n=\sum_{\nu=n}^{\infty}\left(1-\frac{1}{m}\right)^{\nu}$$
$$=m\left(1-\frac{1}{m}\right)^{n},$$

then

$$\sum_{n=m+1}^{\infty}P_n\left(1-\frac{1}{m}\right)^{n}$$
$$=\sum_{n=m+1}^{\infty}P_n(R_n-R_{n+1})$$
$$=P_{m+1}R_{m+1}+\sum_{n=m+2}^{\infty}R_n(P_n-P_{n-1})$$
$$=P_{m+1}m\left(1-\frac{1}{m}\right)^{m+1}+\sum_{n=m+2}^{\infty}p_nR_n$$

from (10). Hence

$$|J|\le\varepsilon M\frac{p_m}{P_m^2}P_{m+1}m\left(1-\frac{1}{m}\right)^{m+1}+$$
$$+\varepsilon M\frac{p_m}{P_m^2}\sum_{n=m+2}^{\infty}p_n m\left(1-\frac{1}{m}\right)^{n}$$
$$=S_1+S_2,\qquad \text{say.}$$

Here we see easily

$$0\le S_1\le\varepsilon M$$

from (10), and further

$$0\le S_2\le\varepsilon M\frac{p_m^2}{P_m^2}m\sum_{n=m+2}^{\infty}\left(1-\frac{1}{m}\right)^{n}$$
$$\le\varepsilon M\left(\frac{p_m}{P_m}m\right)^{2}$$
$$\le\varepsilon M$$

again from (10). Therefore we have

$$|J|\le\varepsilon M$$

for sufficiently large m.

Hence, letting m increase indefinitely, we have

$$\lim_{m\to\infty}s_m=\lim_{x\to1-0}\frac{p_s(x)}{p(x)}=s,$$

which proves the theorem.

Finally I wish to express my hearty thanks to Professor G. Brauer for his kind conjecture on Theorem 2.

References

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