31. *Approximative Dimension of a Space of Analytic Functions*

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(Comm. by Kinjirô KUNUGI, M.J.A., Feb. 12, 1965)

Banach [1] introduced the concept of linear dimension into the theory of topological linear spaces. Extending the idea, Kolmogorov [2] defined the *approximative dimension* with a view to more definite comparison between dimensions of certain linear spaces. Besides the definition he gave some of its examples in the note [2]. Among them we find a formula determining the approximative dimension for the space A_G^s of regular analytic functions defined on a domain G of s complex variables, with which the comparison of dimensions for different s leads to a reasonable result. The proof is not given, only it is mentioned that the formula can be derived by the same method as is used for the evaluation of ε-entropies. But A_G^s with the topology considered here being countably normed, i.e. not having such a simple metric as is usually taken to define ε-entropies, the circumstances are somewhat more complicated, the proof of the formula seems by no means trivial.

The purpose of the present paper is to give a complete proof to the formula in the simplest case where $s=1$ and $G=\{z: |z|<1\}$. In the proof we use some results in the theory of ε-entropies [3]. For general s, because those results are also available, the proof given here remains unchanged in essentials, as long as G is suitably simple so that it can be reduced to a polycylinder.

DEFINITION (Kolmogorov). To every topological linear space E we assign such a family $\Phi(E)$ of functions $\varphi(\varepsilon)$ defined for $\varepsilon>0$ as follows. A function $\varphi(\varepsilon)$ belongs to $\Phi(E)$ if and only if for every compact $K\subset E$ and every open neighborhood U of zero in E there exists a positive number ε_0 such that, when $\varepsilon<\varepsilon_0$, we can find $N\leqq\varphi(\varepsilon)$ points $x_1, \cdots, x_N$ in E forming a ε-net of K relative to U, i.e.

$$K\subset \bigcup_{i=1}^{N} (x_i+\varepsilon U).$$

The family $\Phi(E)$ is called the *approximative dimension* of E.[1]

Now let A_G be the space of regular functions on the open disk

1) In Kolmogorov's original definition, the approximative dimension $d_a(E)$ of E is not the family $\Phi(E)$ itself, but defined by comparison: for two topological linear spaces E and E', $d_a(E)\leqq d_a(E')$ if and only if $\Phi(E)\supset\Phi(E')$.

$G=\{z: |z|<1\}$ with the topology of uniform convergence on every compact subset of G. We shall write also $r_n=1-\frac{1}{n}$, $G_n=\{z: |z|<r_n\}$ and $\bar{G}_n=\{z: |z|\leqq r_n\}$ for $n=2, 3, \cdots$. We can take as a local base of A_G the family of neighborhoods $\{\varepsilon_\nu U_n: \nu=1, 2, \cdots; n=2, 3, \cdots\}$, where $\varepsilon_\nu \downarrow 0$ and

$$U_n=\{f(z)\in A_G: \sup_{z\in\bar{G}_n} |f(z)|<1\}.$$

THEOREM (Kolmogorov). *A function* $\varphi(\varepsilon)$ *for* $\varepsilon>0$ *belongs to* $\Phi(A_G)$ *if and only if*

$$\lim_{\varepsilon\to 0}\frac{\log\varphi(\varepsilon)}{\left(\log\frac{1}{\varepsilon}\right)^2}=\infty.$$

Proof. Suppose that we are given an arbitrary compact subset $K\subset A_G$ and a neighborhood U of zero in A_G. We can choose a sufficiently small neighborhood $\varepsilon_\nu U_n$ from among elements of the local base taken above such that its closure in A_G

$$\varepsilon_\nu \bar{U}_n=\{f(z)\in A_G: \sup_{z\in\bar{G}_n} |f(z)|\leqq\varepsilon_\nu\}\subset U. \tag{1}$$

Take an integer $m>n$, and put

$$\sup_{f\in K}\sup_{z\in\bar{G}_m} |f(z)|=C, \tag{2}$$

the left hand side of (2) being finite because of compactness of K.

Let $A_{G_m}^{\bar{G}_n}(C)$ be the space of regular functions $f(z)$ on G_m not exceeding C by moduli with the uniform metric on $\bar{G}_n\subset G_m$, i.e.

$$A_{G_m}^{\bar{G}_n}(C)=\{\text{regular } f(z) \text{ on } G_m: \sup_{z\in G_m} |f(z)|\leqq C\},$$

$$\rho_n(f_1, f_2)=\sup_{z\in\bar{G}_n} |f_1(z)-f_2(z)| \text{ for } f_1, f_2\in A_{G_m}^{\bar{G}_n}(C).$$

If we introduce the same metric ρ_n (i.e. the uniform metric on $\bar{G}_n$) into the spaces R and $\tilde{A}$ of all regular functions on G_m and G respectively (A_G and $\tilde{A}$ are the same as point set, but with different topologies), $A_{G_m}^{\bar{G}_n}(C)$ and $\tilde{A}$ are subspaces of R, and it holds

$$K\subset\tilde{A}\cap A_{G_m}^{\bar{G}_n}(C)\subset R.$$

The set K considered with the metric ρ_n as a subspace of R will be denoted for a while by the same K without any confusion. $\bar{U}_n$ is the unit sphere of the space $\tilde{A}$.

Now, given a positive number ε, we consider the most economical $\varepsilon\varepsilon_\nu$-net $\{x_1, \cdots, x_N\}$ of K in the metric space $\tilde{A}$, i.e. $x_1, \cdots, x_N\in\tilde{A}$ and

$$K\subset\bigcup_{i=1}^{N}(x_i+\varepsilon\varepsilon_\nu\bar{U}_n), \tag{3}$$

the number N of points being minimal under the above condition. By definition, the logarithm of N to the base 2 is the $\varepsilon\varepsilon_\nu$-entropy of the set K relative to $\tilde{A}$:

$$(4)\qquad \log_2 N = H^{\tilde{A}}_{\varepsilon\varepsilon_\nu}(K).$$

We introduce here some more quantities of this kind: $\varepsilon\varepsilon_\nu$-capacities $C_{\varepsilon\varepsilon_\nu}(K)$, $C_{\varepsilon\varepsilon_\nu}(A^{\bar{G}_n}_{\bar{G}_m}(C))$, and $\varepsilon\varepsilon_\nu$-entropy $H_{\varepsilon\varepsilon_\nu}(A^{\bar{G}_n}_{\bar{G}_m}(C))$. By elementary theorems in the theory of ε-entropies (cf. [3], § 1), and because $K \subset A^{\bar{G}_n}_{\bar{G}_m}(C)$, we have

$$(5)\qquad H^{\tilde{A}}_{\varepsilon\varepsilon_\nu}(K) \leqq C_{\varepsilon\varepsilon_\nu}(K) \leqq C_{\varepsilon\varepsilon_\nu}(A^{\bar{G}_n}_{\bar{G}_m}(C)).$$

For the space $A^{\bar{G}_n}_{\bar{G}_m}(C)$, the asymptotic behavior of its ε-entropy and ε-capacity is known (cf. [3], § 7), that is, for $\varepsilon \to 0$

$$H_\varepsilon(A^{\bar{G}_n}_{\bar{G}_m}(C)) \sim C_\varepsilon(A^{\bar{G}_n}_{\bar{G}_m}(C)) \sim \frac{\left(\log_2 \dfrac{1}{\varepsilon}\right)^2}{\log_2 \dfrac{r_m}{r_n}}.$$

We see that the last side of inequalities (5) has the same order of infinity for $\varepsilon \to 0$:

$$(6)\qquad C_{\varepsilon\varepsilon_\nu}(A^{\bar{G}_n}_{\bar{G}_m}(C)) \sim \frac{\left(\log_2 \dfrac{1}{\varepsilon\varepsilon_\nu}\right)^2}{\log_2 \dfrac{r_m}{r_n}} \sim \frac{\left(\log_2 \dfrac{1}{\varepsilon}\right)^2}{\log_2 \dfrac{r_m}{r_n}}.$$

Now suppose a function $\varphi(\varepsilon)$ of $\varepsilon > 0$ satisfies the condition

$$(7)\qquad \lim_{\varepsilon \to 0} \frac{\log_2 \varphi(\varepsilon)}{\left(\log_2 \dfrac{1}{\varepsilon}\right)^2} = \infty.$$

Then for sufficiently small $\varepsilon < \varepsilon_0$ we have from (4), (5), (6), and (7)

$$\log_2 \varphi(\varepsilon) > C_{\varepsilon\varepsilon_\nu}(A^{\bar{G}_n}_{\bar{G}_m}(C)) \geqq \log_2 N,$$

while from (1) and (3)

$$K \subset \bigcup_{i=1}^{N} (x_i + \varepsilon U).$$

Thus we found a ε-net $\{x_1, \cdots, x_N\} \subset A_G$ of K relative to U, where the number of points $N < \varphi(\varepsilon)$ for $\varepsilon < \varepsilon_0$. We conclude that $\varphi(\varepsilon) \in \Phi(A_G)$.

Conversely, let $\varphi(\varepsilon)$ be an arbitrary element of the family $\Phi(A_G)$. Take a compact subset

$$K = \{\text{regular } f(z) \text{ on } G: \sup_{z \in G} |f(z)| \leqq C\}$$

and a neighborhood $U = U_n$ in A_G for some fixed $C > 0$ and some $n \geqq 2$. Consider the most economical ε-net $\{x_1, \cdots, x_N\} \subset A_G$ of K relative to U, i.e.

$$(8)\qquad K \subset \bigcup_{i=1}^{N} (x_i + \varepsilon U),$$

the number N being minimal under the condition. Now we introduce into the set K the uniform metric ρ_n on $\bar{G}_n$, and denote the obtained space by $A^{\bar{G}_n}_{G}(C)$ in accordance with the previous notation. Let $\tilde{A}$ denote again the space of the set A_G with the metric ρ_n, and $\bar{U}_n \supset U$ its unit sphere. From (8) surely

$$A_G^{\bar{G}_n}(C) \subset \bigcup_{i=1}^{N} (x_i + \varepsilon \bar{U}_n),$$

hence, by the definition of relative ε-entropy,

$$\log_2 N \geqq H_\varepsilon^{\tilde{A}}(A_G^{\bar{G}_n}(C)). \tag{9}$$

Simple facts about ε-entropies as cited above lead to

$$H_\varepsilon^{\tilde{A}}(A_G^{\bar{G}_n}(C)) \geqq H_\varepsilon(A_G^{\bar{G}_n}(C)) \sim \frac{\left(\log_2 \frac{1}{\varepsilon}\right)^2}{\log_2 \frac{1}{r_n}} \text{ for } \varepsilon \to 0. \tag{10}$$

Because $\varphi(\varepsilon) \in \Phi(A_G)$, there exists a positive number ε_0 such that if $\varepsilon < \varepsilon_0$, $N \leqq \varphi(\varepsilon)$. This means that, whenever $\varepsilon < \varepsilon_0$, it holds

$$\frac{\log_2 \varphi(\varepsilon)}{\left(\log_2 \frac{1}{\varepsilon}\right)^2} \geqq \frac{\log_2 N}{\left(\log_2 \frac{1}{\varepsilon}\right)^2} \geqq \frac{H_\varepsilon(A_G^{\bar{G}_n}(C))}{\left(\log_2 \frac{1}{\varepsilon}\right)^2}. \tag{11}$$

From (10) the last side of inequalities (11) tends to the limit $1/\log_2 \frac{1}{r_n}$ when $\varepsilon \to 0$. Meanwhile, n being arbitrary, $r_n = 1 - \frac{1}{n}$ can be taken as close to unity as we desire, so that the limit $1/\log_2 \frac{1}{r_n}$ can be arbitrarily large. This fact with the inequalities (11) leads to the result

$$\lim_{\varepsilon \to 0} \frac{\log_2 \varphi(\varepsilon)}{\left(\log_2 \frac{1}{\varepsilon}\right)^2} = . \infty$$

References

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[3] Kolmogorov, A. N., and Tihomirov, V. M.: ε-entropy and ε-capacity of sets in functional spaces (in Russian). Uspehi Mat. Nauk, **14** (2), 3-86 (1959). (1959).