25. On the Covering Dimension of Certain Product Spaces

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In our previous paper [5], we have proved: *If a product space $X\times Y$ of a space X with a separable metric space Y is countably paracompact and normal, then*

$$\dim (X\times Y)\leqq \dim X+\dim Y.$$

Here $\dim X$ means the covering dimension of X.

In the present paper, we shall establish that if X is a normal P-space [1] the above inequality holds for any metric space Y with an open basis which is a countable union of star-finite systems, even if Y is not separable. Here, *a topological space X is called a P-space if for any set Ω of indices and for any family $\{G(\alpha_1, \alpha_2, \cdots, \alpha_i) \mid \alpha_\nu \in \Omega;\ i=1, 2, \cdots\}$ of open subsets of X such that $G(\alpha_1, \cdots, \alpha_i)\subset G(\alpha_1, \cdots, \alpha_i, \alpha_{i+1})$ for $\alpha_\nu\in\Omega$ and $i=1, 2, \cdots$, there is a family $\{F(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu\in\Omega;\ i=1, 2, \cdots\}$ of closed subsets of X such that* (a) *$F(\alpha_1, \cdots, \alpha_i)\subset G(\alpha_1, \cdots, \alpha_i)$ for $\alpha_\nu\in\Omega(\nu=1, \cdots, i)$ and* (b) *$X=\bigcup_{i=1}^{\infty} F(\alpha_1, \cdots, \alpha_i)$ provided that $X=\bigcup_{i=1}^{\infty} G(\alpha_1, \cdots, \alpha_i)$.*

This concept of P-spaces which is weaker than perfect normality and somewhat stronger than countable paracompactness was introduced by K. Morita [1] in his study on the normality of product spaces, and it was established by him that X is a normal P-space if and only if $X\times Y$ is normal for any metric space Y. Thus our assumption imposed upon X may be said to be reasonable. It is to be noted that every separable metric space has always an open basis which is star-finite.

Theorem 1 has been already proved by K. Morita in his unpublished paper, but in this paper we shall give our proof for the sake of completeness and for its own interest.

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1. The following Lemma has been already presented in [5] with more general form.

Lemma. *If* $\dim Y=0$ *for a metric space Y, there are a countable number of open coverings $V_i=\{V_{i\alpha} \mid \alpha\in\Omega_i\}$ $(i=1, 2, \cdots)$ of Y such that* (a) *$V_{i\alpha}$ is open and closed for any i and α,* (b) *$V_{i\alpha}\cap V_{i\beta}=\phi$ provided $\alpha\neq\beta$,* (c) *$\bigcup_i V_i$ is an open basis of Y.*

Theorem 1. *If X is a normal P-space and Y is a metric space such that* $\dim Y=0$, *then*

$$\dim(X\times Y)\leqq\dim X.$$

Proof. Define, with $V_{i\alpha}$ in Lemma,

$$W(\alpha_1,\alpha_2,\cdots,\alpha_i)=V_{1\alpha_1}\cap V_{2\alpha_2}\cap\cdots\cap V_{i\alpha_i}.$$

Suppose that $\dim X=m$, and let $\{U_1, U_2, \cdots, U_k\}$ be an arbitrary finite open covering of $X\times Y$. We shall construct a refinement $\{\tilde{U}_1, \cdots, \tilde{U}_k\}$ with order $\leqq m+1$.

Let us define, for each l: $1\leqq l\leqq k$,

(1) $G_l(\alpha_1,\cdots,\alpha_i)=\cup\{P\mid P\times W(\alpha_1,\cdots,\alpha_i)\subset U_l,\ P \text{ open in } X\}$,

(2) $G(\alpha_1,\cdots,\alpha_i)=\bigcup_{l=1}^{k} G_l(\alpha_1,\cdots,\alpha_i)$.

Then it is easy to see that

(3) (a) $G_l(\alpha_1,\cdots,\alpha_i)\subset G_l(\alpha_1,\cdots,\alpha_i,\alpha_{i+1})$, $G(\alpha_1,\cdots,\alpha_i)\subset G(\alpha_1,\cdots,\alpha_i,\alpha_{i+1})$ and (b) $\{G(\alpha_1,\cdots,\alpha_i)\times W(\alpha_1,\cdots,\alpha_i)\mid\alpha_\nu\in\Omega_\nu\ (1\leqq\nu\leqq i);\ i=1,2,\cdots\}$ is an open covering of $X\times Y$.

X being a normal P-space, we can find a family $\{F(\alpha_1,\cdots,\alpha_i)\mid \alpha_\nu\in\Omega_\nu\ (1\leqq\nu\leqq i);\ i=1,2,\cdots\}$ of closed subsets of X such that

(4) (a) $F(\alpha_1,\cdots,\alpha_i)\subset G(\alpha_1,\cdots,\alpha_i)$, (b) $X=\bigcup_{i=1}^{\infty}F(\alpha_1,\cdots,\alpha_i)$ provided $X=\bigcup_{i=1}^{\infty}G(\alpha_1,\cdots,\alpha_i)$.

In fact, setting $\Omega=\bigcup_i\Omega_i$, consider all sequences $(\alpha_1,\cdots,\alpha_i)$ in $\Omega (i=1,2,\cdots)$, and let $\tilde{G}(\alpha_1,\cdots,\alpha_i)=G(\alpha_1,\cdots,\alpha_i)$ provided $\alpha_\nu\in\Omega_\nu$ $(1\leqq\nu\leqq i)$ and $\tilde{G}(\alpha_1,\cdots,\alpha_i)=X$ otherwise. Then, for $\alpha_\nu\in\Omega(1\leqq\nu\leqq i+1)$, we have $\tilde{G}(\alpha_1,\cdots,\alpha_i)\subset\tilde{G}(\alpha_1,\cdots,\alpha_i,\alpha_{i+1})$; and thus, by the definition of P-spaces, there is a family $\{\tilde{F}(\alpha_1,\cdots,\alpha_i)\mid\alpha_\nu\in\Omega(1\leqq\nu\leqq i);\ i=1,2,\cdots\}$ of closed subsets of X such that $\tilde{F}(\alpha_1,\cdots,\alpha_i)\subset\tilde{G}(\alpha_1,\cdots,\alpha_i)$ and $X=\bigcup_{i=1}^{\infty}\tilde{F}(\alpha_1,\cdots,\alpha_i)$ provided $X=\bigcup_{i=1}^{\infty}\tilde{G}(\alpha_1,\cdots,\alpha_i)$. Let us put $F(\alpha_1,\cdots,\alpha_i)=\tilde{F}(\alpha_1,\cdots,\alpha_i)$ for such sequences $(\alpha_1,\cdots,\alpha_i)$ as $\alpha_\nu\in\Omega_\nu$ $(1\leqq\nu\leqq i)$, then $\{F(\alpha_1,\cdots,\alpha_i)\}$ satisfies (a) and (b) of (4).

Now, from (3)(b) and (4), it can be easily shown that

(5) $\{F(\alpha_1,\cdots,\alpha_i)\times W(\alpha_1,\cdots,\alpha_i)\mid\alpha_\nu\in\Omega_\nu\ (1\leqq\nu\leqq i);\ i=1,2,\cdots\}$ is a covering of $X\times Y$.

Owing to (2), (4) and the normality of X there are closed sets $F_l(\alpha_1,\cdots,\alpha_i)$ of X such that

(6) (a) $F(\alpha_1,\cdots,\alpha_i)=\bigcup_{l=1}^{k}F_l(\alpha_1,\cdots,\alpha_i)$, (b) $F_l(\alpha_1,\cdots,\alpha_i)\subset G_l(\alpha_1,\cdots,\alpha_i)$ for $1\leqq l\leqq k$.

By (5) and (6), $\{F_l(\alpha_1,\cdots,\alpha_i)\times W(\alpha_1,\cdots,\alpha_i)\mid\alpha_\nu\in\Omega_\nu\ (1\leqq\nu\leqq i);\ i=1,2,\cdots;\ 1\leqq l\leqq k\}$ turns out to be a covering of $X\times Y$.

Since, for any $\alpha_1\in\Omega_1$, $\boldsymbol{U}(\alpha_1)=\left\{G_l(\alpha_1),\ X-\bigcup_{\lambda=1}^{k}F_\lambda(\alpha_1)\mid 1\leqq l\leqq k\right\}$ is an open covering of X, there are two open refinements $\boldsymbol{H}(\alpha_1)=\{H_l(\alpha_1), H_0(\alpha_1)\mid 1\leqq l\leqq k\}$ and $\boldsymbol{R}(\alpha_1)=\{R_l(\alpha_1), R_0(\alpha_1)\mid 1\leqq l\leqq k\}$ such that

(7) (a) $G_l(\alpha_1)\supset H_l(\alpha_1)\supset\overline{R_l(\alpha_1)}$ $(0\leqq l\leqq k)$, (b) the order of $\{H_l(\alpha_1)\mid 0\leqq l\leqq k\}\leqq m+1$, where $G_0(\alpha_1)=X-\bigcup_{\lambda=1}^{k}F_\lambda(\alpha_1)$.

Then we obtain easily

(8) (a) $\bigcup_{l=1}^{k}R_l(\alpha_1)\supset\bigcup_{l=1}^{k}F_l(\alpha_1)$, (b) the order of $\{\overline{R_l(\alpha_1)}\mid 1\leqq l\leqq k\}\leqq m+1$.

In general, if $\alpha_\nu\in\Omega_\nu$ we can show that there are open sets $H_l(\alpha_1,\cdots,\alpha_i)$ and $R_l(\alpha_1,\cdots,\alpha_i)$ such that

(9) (a) $G_l(\alpha_1,\cdots,\alpha_i)\supset H_l(\alpha_1,\cdots,\alpha_i)\supset\overline{R_l(\alpha_1,\cdots,\alpha_i)}$, (b) $\bigcup_{l=1}^{k}R_l(\alpha_1,\cdots,\alpha_i)\supset\bigcup_{l=1}^{k}F_l(\alpha_1,\cdots,\alpha_i)$ and (c) the order of $\{H_l(\alpha_1,\cdots,\alpha_i)\mid 1\leqq l\leqq k\}\leqq m+1$.

The proof is carried out by an induction.

For $i=1$, (7) and (8) are no more than (9). Now assuming the existence of $H_l(\alpha_1,\cdots,\alpha_j)$ and $R_l(\alpha_1,\cdots,\alpha_j)$ which satisfy (9) for any $j<i$ $(i\geqq 2)$, we shall show the existence of them for i.

We put

(10) $G'_l(\alpha_1,\cdots,\alpha_i)=G_l(\alpha_1,\cdots,\alpha_i)-\bigcup_{l=1}^{k}\overline{R_l(\alpha_1,\cdots,\alpha_{i-1})}$.

Evidently, $\boldsymbol{U}(\alpha_1,\cdots,\alpha_i)=\left\{G'_l(\alpha_1,\cdots,\alpha_i),\ H_l(\alpha_1,\cdots,\alpha_{i-1}),\ X-\bigcup_{\lambda=1}^{k}F_\lambda(\alpha_1,\cdots,\alpha_i)\mid 1\leqq l\leqq k\right\}$ is an open covering of X, hence there are two open refinements $\boldsymbol{H}(\alpha_1,\cdots,\alpha_i)=\{H'_l(\alpha_1,\cdots,\alpha_i),H''_l(\alpha_1,\cdots,\alpha_i),H_0(\alpha_1,\cdots,\alpha_i)\mid 1\leqq l\leqq k\}$ and $\boldsymbol{R}(\alpha_1,\cdots,\alpha_i)=\{R'_l(\alpha_1,\cdots,\alpha_i),R''_l(\alpha_1,\cdots,\alpha_i),R_0(\alpha_1,\cdots,\alpha_i)\mid 1\leqq l\leqq k\}$ such that

(11) (a) $G'_l(\alpha_1,\cdots,\alpha_i)\supset H'_l(\alpha_1,\cdots,\alpha_i)\supset\overline{R'_l(\alpha_1,\cdots,\alpha_i)}$,

(b) $H_l(\alpha_1,\cdots,\alpha_{i-1})\supset H''_l(\alpha_1,\cdots,\alpha_i)\supset\overline{R''_l(\alpha_1,\cdots,\alpha_i)}$,

(c) $X-\bigcup_{\nu=1}^{k}F_\nu(\alpha_1,\cdots,\alpha_i)\supset H_0(\alpha_1,\cdots,\alpha_i)\supset\overline{R_0(\alpha_1,\cdots,\alpha_i)}$,

(d) the order of $\boldsymbol{H}(\alpha_1,\cdots,\alpha_i)\leqq m+1$.

If we put

(12) $R_l(\alpha_1,\cdots,\alpha_i)=R_l(\alpha_1,\cdots,\alpha_{i-1})\cup R'_l(\alpha_1,\cdots,\alpha_i)\cup R''_l(\alpha_1,\cdots,\alpha_i)$

we have:

(13) (a) $\bigcup_{l=1}^{k}R_l(\alpha_1,\cdots,\alpha_i)\supset\bigcup_{l=1}^{k}F_l(\alpha_1,\cdots,\alpha_i)$,

(b) the order of $\{\overline{R_l(\alpha_1,\cdots,\alpha_i)}\mid 1\leqq l\leqq k\}\leqq m+1$.

It suffices to show (13)(b). Hence assuming, for instance,

(14) $\bigcap_{l=1}^{m+2}\overline{R_l(\alpha_1,\cdots,\alpha_i)}\neq\phi$

we shall indicate that it reduces to a contradiction.

By (12), $\bigcap_{l=1}^{m+2}\left[\overline{R_l(\alpha_1,\cdots,\alpha_{i-1})}\cup\overline{R'_l(\alpha_1,\cdots,\alpha_i)}\cup\overline{R''_l(\alpha_1,\cdots,\alpha_i)}\right]\neq\phi$, hence it follows that

$$\begin{aligned}(15)\qquad \bigcup\Big[&(\overline{R_{r_1}(\alpha_1,\cdots,\alpha_{i-1})}\cap\cdots\cap\overline{R_{r_\lambda}(\alpha_1,\cdots,\alpha_{i-1})})\\ &\cap\left((\overline{R'_{s_1}(\alpha_1,\cdots,\alpha_i)}\cap\cdots\cap\overline{R'_{s_\mu}(\alpha_1,\cdots,\alpha_i)})\right.\\ &\cap\left((\overline{R''_{t_1}(\alpha_1,\cdots,\alpha_i)}\cap\cdots\cap\overline{R''_{t_\nu}(\alpha_1,\cdots,\alpha_i)})\right.\Big]\neq\phi\end{aligned}$$

where $(r_1, \cdots, r_\lambda, s_1, \cdots, s_\mu, t_1, \cdots, t_\nu)$ ranges over all permutations of $(1, 2, \cdots, m+2)$. Since $\overline{R_l(\alpha_1, \cdots, \alpha_{i-1})} \cap \overline{R'_h(\alpha_1, \cdots, \alpha_i)} = \phi$ by virtue of (10) and (11)(a), we have either $\lambda=0$ or $\mu=0$ for non-empty terms in (15). In the case where $\lambda=0$, (15) reduces to $\overline{(R'_{s_1}(\alpha_1, \cdots, \alpha_i)} \cap \cdots \cap \overline{R'_{s_\mu}(\alpha_1, \cdots, \alpha_i))} \cap \left(\overline{(R''_{t_1}(\alpha_1, \cdots, \alpha_i)} \cap \cdots \cap \overline{R''_{t_\nu}(\alpha_1, \cdots, \alpha_i))} \neq \phi\right.$, and this contradicts (11)(d). If $\mu=0$ (15) reduces to $\overline{(R_{r_1}(\alpha_1, \cdots, \alpha_{i-1})} \cap \cdots \cap \overline{R_{r_\lambda}(\alpha_1, \cdots, \alpha_{i-1}))} \cap \left(\overline{(R''_{t_1}(\alpha_1, \cdots, \alpha_i)} \cap \cdots \cap \overline{R''_{t_\nu}(\alpha_1, \cdots, \alpha_i))} \neq \phi\right.$ and thus we have, by the assumption of our induction and (11)(b), $H_{r_1}(\alpha_1, \cdots, \alpha_{i-1}) \cap \cdots \cap H_{r_\lambda}(\alpha_1, \cdots, \alpha_{i-1}) \cap H_{t_1}(\alpha_1, \cdots, \alpha_{i-1}) \cap \cdots \cap H_{t_\nu}(\alpha_1, \cdots, \alpha_{i-1}) \neq \phi$ which contradicts (9)(c) for $i-1$. And (13)(b) follows.

Now $G_l(\alpha_1, \cdots, \alpha_i) \supset \overline{R_l(\alpha_1, \cdots, \alpha_i)}$ is evident. Thus we have an open sets family $\{H_l(\alpha_1, \cdots, \alpha_i) \mid 1 \leqq l \leqq k\}$ such that

(16) (a) $G_l(\alpha_1, \cdots, \alpha_i) \supset \overline{H_l(\alpha_1, \cdots, \alpha_i)} \supset H_l(\alpha_1, \cdots, \alpha_i) \supset \overline{R_l(\alpha_1, \cdots, \alpha_i)}$,

(b) $\{H_l(\alpha_1, \cdots, \alpha_i) \mid 1 \leqq l \leqq k\}$ is similar to $\{\overline{R_l(\alpha_1, \cdots, \alpha_i)} \mid 1 \leqq l \leqq k\}$.

Clearly, from (13)(b) and (16)(b), it follows that

(17) the order of $\{H_l(\alpha_1, \cdots, \alpha_i) \mid 1 \leqq l \leqq k\} \leqq m+1$.

(13), (16)(a) and (17) show that $H_l(\alpha_1, \cdots, \alpha_i)$ and $R_l(\alpha_1, \cdots, \alpha_i)$ satisfy (9) for i, and the induction completes.

Let us put

$\tilde{U}_l = \cup \{R_l(\alpha_1, \cdots, \alpha_i) \times W(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu \in \Omega_\nu;\ i=1, 2, \cdots\}$ then $\tilde{U}_l \subset U_l$ and the order of $\{\tilde{U}_1, \tilde{U}_2, \cdots, \tilde{U}_k\}$ is at most $m+1$.

In fact, if $\bigcap_{\lambda=1}^{m+2} \tilde{U}_{r_\lambda} \neq \phi$, then we have $\bigcap_{\lambda=1}^{m+2} \{R_{r_\lambda}(\alpha_1, \cdots, \alpha_{i_\lambda}) \times W(\alpha_1, \cdots, \alpha_{i_\lambda}) \mid \alpha_\nu \in \Omega_\nu\} \neq \phi$ for some $(i_1, i_2, \cdots, i_{m+2})$, where we may assume $i_1 \leqq i_2 \leqq \cdots \leqq i_{m+2}$; because, owing to (a) of Lemma, $W(\alpha_1, \cdots, \alpha_i) \cap W(\beta_1, \cdots, \beta_j) \neq \phi$ implies $\beta_\nu = \alpha_\nu$, $\nu = 1, 2, \cdots, i$ for $i \leqq j$. On the other hand, $R_l(\alpha_1, \cdots, \alpha_i) \supset R_l(\alpha_1, \cdots, \alpha_{i-1})$ follows from (12). Thus $\bigcap_{\lambda=1}^{m+2} R_{r_\lambda}(\alpha_1, \cdots, \alpha_{i_{m+2}}) \neq \phi$. By virtue of (13)(b), we arrive at a contradiction. q.e.d.

We will define a metric space which will be needed later.

For any two sequences of elements from a non-empty set Ω: $\alpha = (\alpha_1, \alpha_2, \cdots)$, $\beta = (\beta_1, \beta_2, \cdots)$, $\alpha_i, \beta_j \in \Omega$, we define the metric $\rho(\alpha, \beta)$ as follows.

$\rho(\alpha, \beta) = \frac{1}{k}$ if $\alpha_i = \beta_i$ for $i < k$ and $\alpha_k \neq \beta_k$; $\rho(\alpha, \alpha) = 0$. Then the set of all sequences of elements from Ω turns out to be a metric space with $\rho(\alpha, \beta)$ as its metric. We shall denote this space by $N(\Omega)$ (see [2]). Since $\dim N(\Omega) = 0$ ([2, p. 361]). We have

Corollary. *Let X be a normal P-space; then*

$$\dim (X \times N(\Omega)) \leqq \dim X.$$

2. The following theorem is well known.

Theorem 2. *Let X be a normal space and S its F_σ-subset;*

then $\dim S \leqq \dim X$.

Theorem 3. *Let X be a normal P-space and Z a metric space with Y as its subspace; then*

$$\dim (X\times Y) \leqq \dim (X\times Z).$$

Proof. Suppose that $\dim(X\times Z)=s$. Let $\{V_1, V_2, \cdots, V_k\}$ be an arbitrary finite open covering of $X\times Y$ and let $k>s+1$. We can express: $V_i=U_i\cap(X\times Y)$, where U_i is some open subset of $X\times Z$. Put $U=\bigcup_{i=1}^{k} U_i$.

Define open subsets $G_l(\alpha_1, \cdots, \alpha_i)$ of X quite analogously to (1) in the proof of Theorem 1; then $G_l(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i)\subset U_l$ and $G_l(\alpha_1, \cdots, \alpha_i)\subset G_l(\alpha_1, \cdots, \alpha_i, \alpha_{i+1})$, where $W(\alpha_1, \cdots, \alpha_i)$ are also defined as before,*) but, instead of subsets of Y, they are subsets of Z at present.

Let us put

$$G(\alpha_1, \cdots, \alpha_i)=\bigcup_{l=1}^{k} G_l(\alpha_1, \cdots, \alpha_i);$$

then $G(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i)\subset U=\bigcup_{i=1}^{k} U_i$.

Since X is a P-space, by the analogous argument to the proof of Theorem 1, there is a family $\{F(\alpha_1, \cdots, \alpha_i)\mid \alpha_\nu\in\Omega_\nu;\ i=1, 2, \cdots\}$ of closed sets such that (a) $F(\alpha_1, \cdots, \alpha_i)\subset G(\alpha_1, \cdots, \alpha_i)$ and (b) $X=\bigcup_{i=1}^{\infty} F(\alpha_1, \cdots, \alpha_i)$ provided $X=\bigcup_{i=1}^{\infty} G(\alpha_1,\cdots,\alpha_i)$. Then it is easily shown that $\{F(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i)\mid \alpha_\nu\in\Omega_\nu;\ i=1, 2, \cdots\}$ covers $X\times Y$. (It is to be remarked that $X\times Y=U\cap(X\times Y)$.) $W(\alpha_1, \cdots, \alpha_i)$ being an F_σ-set of Z, we can set:

$F(\alpha_1,\cdots,\alpha_i)\times W(\alpha_1,\cdots,\alpha_i)=\bigcup_{t=1}^{\infty}(F(\alpha_1,\cdots,\alpha_i)\times T_t(\alpha_1,\cdots,\alpha_i))$ where $T_t(\alpha_1, \cdots, \alpha_i)$ is a closed subset of Z for every t. Then

$$(18)\qquad \cup\{F(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i)\mid \alpha_\nu\in\Omega_\nu;\ i=1, 2, \cdots\}$$
$$=\bigcup_{t=1}^{\infty}\bigcup_{i=1}^{\infty}[\cup\{F(\alpha_1, \cdots, \alpha_i)\times T_t(\alpha_1, \cdots, \alpha_i)\mid \alpha_\nu\in\Omega_\nu\}].$$

Now $\cup\{F(\alpha_1, \cdots, \alpha_i)\times T_t(\alpha_1, \cdots, \alpha_i)\mid \alpha_\nu\in\Omega_\nu\}$, which lies in the bracket of (18), is a locally finite union of closed sets, and hence it is closed. Then the left hand side of (18) turns out to be an F_σ-subset of $X\times Z$. And by Theorem 2,

$$\dim[\cup\{F(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i)\mid \alpha_\nu\in\Omega_\nu;\ i=1, 2, \cdots\}]$$
$$\leqq \dim(X\times Z)=s.$$

Let $F=\cup\{F(\alpha_1,\cdots,\alpha_i)\times W(\alpha_1,\cdots,\alpha_i)\mid \alpha_\nu\in\Omega_\nu;\ i=1,2,\cdots\}$; then $X\times Y\subset F\subset U$ and $\dim F\leqq s$. Since $\{F\cap U_l\mid 1\leqq l\leqq k\}$ is an open covering of the subspace F, there exists its open refinement $\{F\cap R_l\mid 1\leqq l\leqq k;\ R_l$ open in $X\times Z\}$ such that $F\cap U_l\supset F\cap R_l$ and the order of $\{F\cap R_l\mid 1\leqq l\leqq k\}\leqq s+1$. Hence, the order of $\{R_l\cap(X\times Y)\mid 1\leqq l\leqq k\}\leqq$

*) Here $\dim Z$ is not necessarily zero; therefore we should employ those V_i which are described in [5, Lemma 1].

$s+1$, $R_l \cap (X\times Y) \subset R_l \cap F \subset U_l \cap F$ and $\{R_l \cap (X\times Y) \mid 1 \leqq l \leqq k\}$ is an open covering of $X\times Y$. Thus, as a covering of $X\times Y$, $\{R_l \cap (X\times Y) \mid 1 \leqq l \leqq k\}$ is an open refinement of $\{U_l \cap (X\times Y) \mid 1 \leqq l \leqq k\}$ whose order is at most $s+1$. Since $U_l \cap (X\times Y) = V_l$, the proof is completed.

3. Now we have our main theorem.

Theorem 4. *Let X be a normal P-space. If Y is a metric space with an open basis which is a countable union of star-finite systems, then*

$$\dim (X\times Y) \leqq \dim X + \dim Y.$$

Proof. If $\dim Y = n$, Y can be considered as a subset of $N(\Omega)\times M$, where M is such a subset of unit $(2n+1)$-cube that at most n of its coordinates are rational ([4]) and $\dim M = n$.

Since $(X\times N(\Omega))$ is a normal P-space ([1, Theorem 4.1]), $(X\times N(\Omega))\times M$ turns out to be countably paracompact and normal ([3, Theorem 2.2]), therefore by [5]

$$\dim ((X\times N(\Omega))\times M) \leqq \dim (X\times N(\Omega)) + \dim M,$$

and hence, by Theorem 1, we have

$$\dim ((X\times N(\Omega))\times M) \leqq \dim X + \dim M.$$

According to Theorem 3,

$$\dim (X\times Y) \leqq \dim (X\times (N(\Omega)\times M)),$$

therefore $\dim (X\times Y) \leqq \dim X + \dim M = \dim X + \dim Y$. q.e.d.

Corollary. *Let X be a normal P-space. If Y is a metric space with the star-finite property, then*

$$\dim (X\times Y) \leqq \dim X + \dim Y.$$

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