97. On Axiom Systems of Propositional Calculi. I

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In a series of our papers, we are mainly concerned with axiom systems of propositional calculi containing many valued propositional calculi and modal logic etc. Our first paper is a preliminary note for our discussions and contains some elementary remarks. In our first some papers, we shall treat two valued propositional calculus.

In this series, we shall use the well known symbolisms formulated by J. Lukasiewicz (see [1] and [2]), inasmuch as Lukasiewicz symbols are quite helpful and useful for our discussions in axiom systems of propositional calculi.

The small Latin letters denote the propositional variables, and the Greek letters denote theses derived from some given axiom system of propositional calculus. The capital Latin letters N and C denote negation and implication respectively.

To prove theses of a system of two valued propositional calculus, we use two fundamental rules of inference.

1) The rule of *substitution*. For a thesis α in a system, the expression β obtained by a correct substitution in α is also a thesis of the system.

2) The rule of *detachment* (modus ponens). If $C\alpha\beta$ and α are theses in the system, then β is a thesis in it.

Of course, $C\alpha\beta$ denotes that α implies β.

In our discussion, we take up the well known system of axioms: $CCpqCCqrCpr$, $CCNppp$, and $CpCNpq$ formulated by J. Lukasiewicz [1] as fundamental one. In his Elements [1], J. Lukasiewicz has proved all axioms in Frege, Russell and Hilbert systems (see the table below) from his axioms using the rules of substitution and detachment. Further J. Lukasiewicz states in his Elements [1] that his first system of axioms are modified by $CCpqCCqrCpr$, $CpCNpq$, and $CCNpqCCqpp$. The proof of the equivalence has not given in his Elements [1]. A simple proof of the equivalence is given by Y. Arai, one of our colleagues, and will be published in the second note of our papers.

J. Lukasiewicz has also given further two systems of axioms, and B. Sobociński has stated two new systems of axioms. On the other hand, we have some single axiom systems by J. Lukasiewicz and A, Tarski [2] and C. Meredith etc. (see A. N. Prior [3]).

Our first attempt is to give direct and economical proofs of equivalences of these all systems each other. In his Elements [1], J. Lukasiewicz has proved all axioms in the following table except the second axiom *CpCqCrp* in Sobociński systems and the first axiom *CCpqCNqCpr* in Sobociński second system of axioms.

We first state these systems of axioms.

Lukasiewicz (L_1)-system	Frege (F)-system
1 *CCpqCCqrCpr*,	1 *CpCqp*,
2 *CCNppp*,	2 *CCpCqrCCpqCpr*,
3 *CpCNpq*.	3 *CCpqCNqNp*,
	4 *CNNpp*,
	5 *CpNNp*.
Russell (R)-system	Hilbert (H)-system
1 *CpCqp*,	1 *CpCqp*,
2 *CCpqCCqrCpr*,	2 *CCpCqrCqCpr*,
3 *CCpCqrCqCpr*,	3 *CCqrCCpqCpr*,
4 *CNNpp*,	4 *CpCNpq*,
5 *CCpNpNp*,	5 *CCpqCCNpqq*.
6 *CCpNqCqNp*.	
Lukasiewicz (L_2)-system	Lukasiewicz (L_3)-system
1 *CCCpqrCNpr*,	1 *CpCqp*,
2 *CCCpqrCqr*,	2 *CCpCqrCCpqCpr*,
3 *CCNprCCqrCCpqr*.	3 *CCNpNqCqp*.
Sobociński (S_1)-system	Sobociński (S_2)-system
1 *CNpCpq*,	1 *CCpqCNqCpr*,
2 *CpCqCrp*,	2 *CpCqCrp*,
3 *CCNprCCqrCCpqr*.	3 *CCNpqCCpqq*.

We remark that *CpCqCrp*, *CCpqCNqCpr* are derived from the (L_1)-system, i.e. Lukasiewicz axioms (see the table). Each proof given below is mechanically given by the proof line, and numbers of the following two proof lines are the numbers of theses in the Elements [1].

The proof of *CpCqCrp* is:

21 $p/q,\ q/p,\ r/Crp$ *C18.3 $r/p,\ s/r$—a),

a) *CpCqCrp*.

On the other hand, the proof of *CCpqCNqCpr* is

5 $p/Nq,\ s/p$ *C36 $p/q,\ q/r$—b),

b) *CCpqCNqCpr*.

Therefore we have $(L_1) \Rightarrow (S_1)$ and (S_2), so (L_1) implies all axioms in the table.

To complete all proofs, we have to give fifty nine proofs of implications among axiom systems above. To prove some of them, we need the following propositions. Under the rules of substitution and detachment, the following five theses are equivalent.

1′ $CpCqp$.

$1'\ p/CrCsr\ {}^*C1'—2'$,

2′ $CqCrCsr$.

$1'\ p/CqCrCsr,\ q/p\ {}^*C2'—3'$,

3′ $CpCqCrCsr$.

$3'\ q/p—4'$,

4′ $CpCpCrCsr$.

$4'\ p/CpCpCrCsr\ {}^*C4'—C4'—5'$,

5′ $CrCsr$.

$2'\ q/CqCrCsr\ {}^*C2'—5'$.

Hence five theses are equivalent.

The most simple case is (H)$\Rightarrow$(L_1). Using the rules of substitution and detachment, from the (H)-system we have the following theses:

$2\ p/Cqr,\ q/Cpq,\ r/Cpr\ {}^*C3—6$,

6 $CCpqCCqrCpr$,

$2\ r/p\ {}^*C1—7$,

7 $CqCpp$,

$7\ q/CpCqp\ {}^*C1—8$,

8 Cpp,

$5\ q/p\ {}^*C8—9$,

9 $CCNppp$.

The theses 6, 9 and the axiom 4 of (H) are axioms of (L_1)-system. Next we shall give a proof of (H)$\Rightarrow$(S_2). As already shown from $CpCqp$, we have

10 $CqCrCsr$.

From thesis 7, we have the following theses:

$2\ p/q,\ q/p,\ r/Crp\ {}^*C7\ r/p,\ s/r—10$,

11 $CpCqCrp$.

$2\ p/Cpq,\ q/CNpq,\ r/q\ {}^*C5—12$,

12 $CCNpqCCpqq$.

$2\ q/Np,\ r/q\ {}^*C4—13$,

13 $CNpCpq$.

$10\ p/Cpq,\ q/CCqrCpr,\ r/s\ {}^*C2—14$,

14 $CCCCqrCprsCCpqs$.

$3\ q/Cqr,\ r/Csr,\ s/CCsqCpCsr\ {}^*C14\ p/s,\ s/CpCsr—15$,

15 $CCpCqrCCsqCpCsr$.

$15\ p/Nq,\ s/p\ {}^*C13\ p/q,\ q/r—16$,

16 $CCpqCNqCpr$.

In the theses obtained, theses 10, 12, and 16 are (S_2)-axioms. This completes the proof of (H) $\Rightarrow$ (S_2). As an application of theses above, we prove an important thesis: *CCpCpqCpq*.

12 *q/Cpq C*13—17,

17 *CCpCpqCpq*.

Next we shall prove three axiom in (S_1). *CNpCpq* is thesis 9 in the proof of (H) $\Rightarrow$ (S_2). To prove other axioms, we need the thesis 17. The third axiom and the second axiom of (F) (or the second axiom of (L_3)) are proved in a same process, but its proofs are not so easy.

3 *p/s, q/CpCqr, r/CqCpr* **C*2—18,

18 *CCsCpCqrCsCqCpr*.

3 *p/s, q/Cpq, r/CCqrCpr* **C*6—19,

19 *CCsCpqCsCCqrCpr*.

18 *p/s, q/Cqr, r/Cpr, s/CsCpr* **C*19—20,

20 *CCsCpqCCqrCsCpr*.

20 *p/Cpq, q/CpCpr, r/Cpr,*
s/CpCqr **C*6 *s/p*—*C*17 *q/r*—21,

21 *CCpCqrCCpqCpr*.

6 *p/CsCpCqr, q/CsCqCpr,*
r/CqCsCpr **C*6—*C*2 *p/s, r/Cpr*—22,

22 *CCsCpCqrCqCsCpr*.

6 *p/CsCpCqr, q/CqCsCpr,*
r/CqCpCsr **C*22—*C*6 *s/q, p/s, q/p*—23,

23 *CCsCpCqrCqCpCsr*.

6 *p/Np, q/Cpq* **C*10—24,

24 *CCCpqrCNpr*.

20 *p/Cqr, q/Cpr, r/CCNprr, s/Cpq* **C*6—*C*5*q/r*—25,

25 *CCpqCCqrCCNprr*.

23 *p/Cqr, q/CNpr, s/Cpq* **C*24—26,

26 *CCNprCCqrCCpqr*.

Further we have the following thesis:

6 *p/q, q/Cpq* **C*1 *p/q, q/p*—27,

27 *CCCpqrCqr*.

Therefore, thesis 26 is the third axiom of (S_1), and at the same time, we have three axioms of (L_2) from theses 24, 26, and 27.

References

[1] J. Lukasiewicz: Elements of Mathematical Logic (translation from Polish). Oxford (1963).

[2] J. Lukasiewicz und A. Tarski: Untersuchungen über den Aussagenkalkül. C. R. de Varsovie, Cl. III, **23**, 30-50 (1930).

[3] A. N. Prior: Formal Logic. Oxford (1962).