119. *Singular Cut-off Process and Lorentz Properties*

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(Comm. by Kinjirô KUNUGI, M.J.A., Sept. 13, 1965)

§ 1. **Introduction.** Usually it is believed that the ordinary cut-off disturbs the ordinary Lorentz covariance and causality condition defined in [4] pp. 249, 250. But by using a sort of singular cut-off which is alike to one defined in [3] p. 377, we can also avoid these difficulties in a sense. Namely, though these difficulties are too essential to avoid by the construction of $\sum_{i=1}^{n} C_i\varphi(x-x_i)$($\varphi(x)$; usual field function), the suitable use of E.R. ν integral for singular mollifier gives the positive effect for this purpose. This positive effect is one of the notable advantage of this E.R. ν singular cut-off. The direct purpose of the use of ν is the selection of the suitable conditional convergence for the definition of integral. It seems that the suitable interpretation of the change of ν used here is also possible from the consideration of inner structure of elementary particle. On the other hand, we can also avoid these difficulties by the change of the definition of Lorentz covariance and causality, which are likely to represent an approximation from another view point [2] p. 73. But in this article we do not treat this mainly. The materials of this article are arranged as follows: in § 2 E.R. ν integral is defined by A-integral form, in § 3 the definition of ν in the four (or three) dimensional E.R. ν singular cut-off and the positive effect of this singular cut-off to ordinary Lorentz covariance are shown, namely this positive effect is to rewrite the change of the smeared out field function by inhomogeneous Lorentz transform (inner product conserving) to the ordinary Lorentz covariant form (except the change of ν), and in § 4 the positive effect of E.R. ν singular cut-off to ordinary causality condition is given except the change of ν.

§ 2. **E.R. ν integral. Definition 1.** Suppose that the function $f(x)$ defined in the interval $[a, b]$ satisfies the following two conditions:

1) $\int_{[a,b]\cap[x;|f(x)|>n\phi(x)]} n\phi(x)dx$ tends to zero as n tends to ∞, where $\phi(x)$ is locally L^1 positive valued function,

2) there exists a finite limit $\lim_{n\to\infty}\int_a^b [f]_n(x)dx$, where

$$[f]_n(x)=\begin{cases} f(x) & \text{for } x \text{ satisfying the relation } |f(x)|\leqq n\phi(x) \\ 0 & \text{for } x \text{ satisfying the relation } |f(x)|>n\phi(x). \end{cases}$$

Then we say that $f(x)$ is E.R. ν integrable and that the above limit is E.R. ν integral of $f(x)$ (same as one by K. Kunugi etc.) which is denoted by E.R. $\nu \int_a^b f(x)dx$. Here ν points out the measure defined by $\nu(B)=\int_B \phi(x)dx$.

We may also use the locally L^1 non-negative valued function $\phi(x)$ with the property $\{x; f(x)\neq 0\}\subseteq\{x; \phi(x)\neq 0\}$. If $\phi(x)\equiv 1$, E.R. ν integral is equal to E.R. (or A) integral.

From the condition 1), an arbitrary non decreasing non-negative real number's sequence (tending to ∞) can be used instead of $\{n\}$ in this definition. In the following Definition 2, suppose that ν (or ϕ) is the measure (or the function) used in Definition 1.

Definition 2. Suppose that $f(x)$ $(-\infty<x<+\infty)$ satisfy the following three conditions;

1) $\int_{[x;|f(x)|>n\phi(x)]} n\phi(x)dx$ tends to zero as n tends to $\infty (\phi(x)\in L^1)$,
2) $[f]_n(x)$ (by ϕ) is not zero in only a finite interval,
3) there exists a limit $\lim_{n\to\infty}\int_{-\infty}^{+\infty}[f]_n(x)dx$ (defined by ϕ).

Then we say that $f(x)$ is improper E.R. ν integrable and the above limit is improper E.R. ν integral of $f(x)$ denoted by imp. E.R. $\nu \int_{-\infty}^{+\infty} f(x)dx$. By using a sort of locally E.R. ν integrability we can define the more general imp. E.R. ν integral.

The above E.R. ν integral does not give by the change of the measure but of the rule of conditional convergence from E.R. integral (or A integral).

§ 3. Lorentz covariance. Hereafter, let $\vec{x}$ denote the three dimensional vector, x denote $(\vec{x}, t)$, $\varphi(x)$ a field function, and $\rho(x)$ a real valued (smooth) function, $[f(x)]^+\equiv(1/2)\{f(x)+|f(x)|\}$ and $[f(x)]^-\equiv(1/2)\{f(x)-|f(x)|\}$.

Suppose that $U(a, \Lambda)$ is the unitary operator depending on the inhomogeneous Lorentz transform (a, Λ) which is defined in the state vector's space [6] p. 22 (or Von Neumann's direct product space). Furthermore, hereafter we mainly use the convolution by the meaning of [3] p. 377 Def. 1, and it is denoted by $*$.

In [3] p. 380 we have discussed about the Λ inhomogeneous Lorentz covariance, and assert the following

Lemma 1. *$\varphi(x)*\rho(x)$ satisfies the Λ inhomogeneous Lorentz covariance.*

Here we will discuss about the ordinary Lorentz covariance defined in [4] p. 249 permitting the variant of ν (or ϕ) in E.R. ν integral.

It is presumed that this covariance is satisfied for the smeared out field function $\varphi(x)*\rho(x)$ by $\rho(x)$ with the property $\rho(x)\equiv\rho(\Lambda x)$. But since the Λ invariant pseudo function $\rho(x)$ with the finite carrier is only $C\delta(x)$, only $\varphi(x)*C\delta(x)=C\varphi(x)$ satisfies this covariance for usual Lebesgue measure ν. Now, let's determine ν (or ϕ) in E.R. ν integral for the use of the other singular mollifier with the property $\rho(x)\equiv\rho(\Lambda x)$ (or with the imp. E.R. equivalent property $\rho(x)\cong\rho(\Lambda x)$). (Even for the three dimensional E.R. ν singular cut-off, the similar discussions are possible.) Since $\rho(x)$ can be completely described (or described by the meaning of imp. E.R. equivalence) by $\rho_1(r)\equiv\rho(\vec{x}, 0)$ $(|\vec{x}|=r)$ and $\rho_2(t)\equiv\rho(\vec{0}, t)$, let's determine the functions $\rho_1(r)\,(r\geqq 0)$ and $\rho_2(t)$ at the first step.

Lemma 2. *There exists a non-negative number's sequence* $\{a_j\}$ *with the properties* $\sum_{j=1}^{\infty} a_j=1$, $\sum_{j=n+1}^{\infty} a_j=o(1/n)$, $\sum_{j=1}^{\infty} j\, a_j=+\infty$ *and* $\lim_{j\to\infty} ja_j=0$.

Proof. As the positive integer's sequence $\{n_k\}$ with the property $\sum_{k=l}^{\infty} 1/n_k<2/n_l$, choose $\{n_k\}$ with the property $n_k\geqq[\exp k]$. Furthermore construct the following $\{b_n\}$ (from this $\{n_k\}$);

$$b_n=\begin{cases}0 & \text{for } n\neq n_k\\ 1/(kn_k) & \text{for } n=n_k.\end{cases}$$

If $a_n=b_n/(\sum_{n=1}^{\infty} b_n)$, then the sequence $\{a_n\}$ satisfies the above conditions.

From this Lemma 2 and [2] p. 76 Lemma 2, we may assert the following

Lemma 3. *The singular function* $f_1(r)$ *(or* $f_2(t)$*) with the following properties can be constructed* (i) $\{r; f_1(r)\neq 0, r\geqq 0\}\subseteq F_\infty$ (ii) $\int_U [f_1(r)]^+\,dr=+\infty$ *and* $\int_U [f_1(r)]^-\,dr=-\infty$ *hold valid for any open* $U\subset\{r; r\geqq 0\}$, *where* F_∞ *is the set defined in* [3] p. 379.

Since $0<B_1(V)\leqq\|\Lambda(\vec{0}, t_1-t_2)\|_{E^4}/|t_1-t_2|\leqq B_2(V)$ and (or) $0<B_1(V)\leqq\|\Lambda(\vec{x}_1-\vec{x}_2, 0)\|_{E^4}/|\|\vec{x}_1\|-\|\vec{x}_2\||\leqq B_2(V)$ hold valid in the arbitrary bounded open $V\subset\{(\vec{x}, t); |\vec{x}|^2\neq t^2\}$, it follows from this Lemma 3 that the following Lemma 4 is asserted.

Lemma 4. *Suppose that* $f(x)=\begin{cases}f_1(r) & \text{for } x=\Lambda(\vec{x}, 0)\,(|\vec{x}|=r)\\ f_2(t) & \text{for } x=\Lambda(\vec{0}, t).\end{cases}$

Then $f(x)$ *has the properties* $\int_U [f(x)]^+dx=+\infty$ *and* $\int_U [f(x)]^-\,dx=-\infty$ *for any open* $U\subset E^4$. *Here* Λ *is an arbitrary homogeneous Lorentz transform.*

Next let's show the properties of locally L^1 non-negative function's sequences $\{\phi_n(x)\}$ and $\{\tilde{\phi}_n(x)\}$ which are useful to the determination of ν (or ϕ) used here.

Suppose that $g(x)$ is a fixed element in $(\mathfrak{S})$ (or $(\mathfrak{D})$) and that $f(x)$ is the singular function defined in Lemma 4.

Let x_0 denote the time t and $E_{p,k,\nu}$ denote the set defined in [3] p. 379. (i) $\phi_n(x)$ and $\tilde{\phi}_n(x)$ are constant in $[\Pi_{i=0}^3\{x_i; \mu_i/2^n\leqq x_i<(\mu_i+1)/2^n\}]\cap\{x; x=\Lambda(x_1, 0, 0, 0), x_1\in E_{p,k,\nu}\cap F\}$ and $[\Pi_{i=0}^3\{x_i; \mu_i/2^n\leqq$

$x_i<(\mu_i+1)/2^n\}] \cap \{x; x=\Lambda(\vec{0}, t), t \in E_{p,k,\nu} \cap F\}$ $(F=\{x; f(x)\neq 0\})$, where $\mu_i=0, \pm 1, \pm 2, \cdots$. (ii) $\phi_n(x)$ and $\tilde{\phi}_n(x)$ take the values $|f(x)|$ or 0 for arbitrary fixed x. (iii) The sets $\{x; \phi_n(x)>0\}$ and $\{x; \tilde{\phi}_m(x)>0\}$ satisfy the following conditions; the sets in the family $[\{x; \phi_n(x)>0\}, \{x; \tilde{\phi}_m(x)>0\} n, m=1, 2, \cdots]$ are compact and disjoint each other, $\cup_n \{x; \phi_n(x)>0\} \supseteq \{x; f(x)>0\}$ and $\cup_m \{x; \tilde{\phi}_m(x)0\} \supseteq \{x; f(x)>0\}$. (iv) The integral

$$\int_{\{x; f(x)>\phi_n(x)\}} \phi_n(x)dx + \int_{\{x; f(x)<-\tilde{\phi}_n(x)\}} \tilde{\phi}_n(x)dx$$

tends to zero as n tends to ∞. (v) There exists a finite fixed limit with the property $\lim_{n\to\infty} \int [f]_n(x)dx = \int g(x)dx$. Here $[f]_n(x)$ is the function

$$[f]_n(x)=\begin{cases} f(x) & \text{for } x \text{ such that } -\tilde{\phi}_n(x) \leqq f(x) \leqq \phi_n(x) \\ \phi_n(x) & \text{for } x \text{ such that } f(x)>\phi_n(x) \\ -\tilde{\phi}_n(x) & \text{for } x \text{ such that } f(x)<-\tilde{\phi}_n(x). \end{cases}$$

According to Riemann's theorem about the conditionally convergent series, the sequences $\{\phi_n(x)\}$ and $\{\tilde{\phi}_n(x)\}$ can be constructed by the refinement of the mesh in E^4 and by the suitable use of the meet this mesh and the subset of the above F. Then we assert the following

Lemma 5. *There exist the sequences* $\{\phi_n(x)\}$ *and* $\{\tilde{\phi}_n(x)\}$ (*related to* $f(x)$ *in Lemma* 4) *satisfying the above properties* (i)—(v).

For the increasing positive integer's sequence $\{N_n\}$, let's construct the non-negative function $\phi(x)=\sum_{n=1}^{\infty}(\phi_n(x)+\tilde{\phi}_n(x))/N_n$ etc. which is positive in the set F. This $\phi(x)$ can be locally summable for a suitable $\{N_n\}$. By using the measure $\nu(B)=\int_B \phi(x)dx$ we assert the following

Theorem 1. *There exists* ν (x, y, z, t) *such that imp.* E.R. $\nu \int f(x)\psi(x)dx = \int g(x)\psi(x)dx$ *holds valid for fixed* $g(x) \in (\mathfrak{S})$ (*or* $(\mathfrak{D})$) *and for any* $\psi(x) \in (\mathfrak{B})$. *Here* $(\mathfrak{B})$ *is the bounded and* C^∞ *functions' space* [5] II, p. 55.

Now suppose that ν (x, y, z, t) has the property such that imp. E.R.

$$\nu\varphi(x) * f(x) \equiv (1/(2\pi)^{3/2})\Big\{\int (a^+(\vec{k})/\sqrt{2k_0})\Big(\text{imp. E.R. } \nu\int f(x') \exp i\{\vec{k}(\vec{x}-\vec{x}') - k_0(t-t')\}dx'\, d\vec{k}\Big) + \text{conjugate term}\Big\}$$

is equal to $\varphi(x) * g(x)$ for a fixed $g(x) \in (\mathfrak{S})$ (or $(\mathfrak{D})$). Since $f(x') \equiv f(\Lambda x')$ and $dx'=d(\Lambda x')$ (for Lebesgue measure) hold valid, the ordinary Lorentz covariance is satisfied for this imp. E.R. $\nu\varphi(x) * f(x)$ by the following meaning;

$$\begin{aligned}&(\Phi, \text{imp. E.R. } \nu\varphi(x) * f(x)\Psi) \\ &\quad = (U\Phi, U(a, \Lambda)\text{ imp. E.R. } \nu\varphi(x) * f(x) U(a, \Lambda)^{-1} U\Psi) \\ &\quad = (U\Phi, [\text{imp. E.R. } \nu_\Lambda \varphi(\tilde{x}) * f(\tilde{x})]_{\tilde{x}=a+\Lambda x} U\Psi)\end{aligned}$$

([3] p. 380 Lemma 1 Proof), where ν_A is the measure with the property $\nu_A(B)=\int_B \phi(A^{-1}x)dx$. By using the singular function's sequence $\{f_m\}$, we can define the singular cut-off with the same properties as [3] p. 380 Theorem 1. If we use three dimensional E.R. ν singular cut-off ([3] p. 377 Def. 1 [3] p. 380 Theorem 1), the above ordinary Lorentz covariance is also satisfied by the more complicated change of ν. Here $f(x')\equiv f(Ax')$ must also weaken suitably. It seems that the change of ν means the change of the appearance physically.

Example. imp. E.R. $\nu\iint d^2k\rho(k^2)/(k^2+m^2)$; $(k^2=k_0^2-k_1^2)$ becomes finite by this Theorem 1 for suitable ν (or ϕ), where $\rho(x)$ is the singular function defined in Lemma 3.

§ 4. Causality. Definition 3.

$$[(A)\varphi * f^{(1)}, (A)\pi * f^{(2)}]$$
$$\equiv(-1/(2\pi)^3)\Big\{i(A)\int f^{(1)}(y)\iint[a(\vec{k}), a^+(\vec{k}')]/2\cdot(A)\int \exp i(\vec{k}(\vec{x}-\vec{y})$$
$$-\vec{k}'(\vec{x}'-\vec{y}')-k_0(t-\tau)+k_0'(t'-\tau')f^{(2)}(y')dy'd\vec{k}\,d\vec{k}'dy+i(A)\int f^{(1)}(y)$$
$$\times\iint[a(\vec{k}), a^+(\vec{k}')]/2\cdot(A)\int\exp(-i)(\vec{k}(\vec{x}-\vec{y})-\vec{k}'(\vec{x}'-\vec{y}')$$
$$-k_0(t-\tau)+k_0'(t'-\tau'))f^{(2)}(y')dy'd\vec{k}\,d\vec{k}'dy\Big\},\ \text{where } y=(\vec{y},\tau),\ y'=(\vec{y}',\tau').$$

We may use various singular integrals instead of A integral in the above Def. 3. Furthermore we can easily apply this definition to three dimensional singular cut-off. Now when the following two conditions are satisfied, we understood that a sort of causality condition was satisfied [2] p. 73;

(1) $\{\vec{x};\rho(\vec{x})\neq 0\}$ is the global (local or generalized) causal set [2] p. 74, where $\rho(\vec{x})$ is three dimensional mollifier used for singular cut-off,

(2) [Boundary $\{\vec{x};\rho(\vec{x})\neq 0\}]=\overline{\{\vec{x};\rho(\vec{x})\neq 0\}}\cap\overline{\{\vec{x};\rho(\vec{x})=0\}}\supseteq\{x;\rho(x)\neq 0\}$.

The ordinary causality condition treated here is the following.

Definition 4. If $[\varphi * f^{(1)}, \pi * f^{(2)}](\tilde{x}-\tilde{x}')=0$ (by the suitable singular integral) holds valid for the two space-like points $\tilde{x}\,\tilde{x}'$, we say that the ordinary causality condition is satisfied by this singular cut-off.

By the same argument as Theorem 1 we can choose $\nu^{(1)}(B)=\int_B \phi^{(1)}(x)dx$ and $\nu^{(2)}(B)=\int_B\phi^{(2)}(x)dx$ such that the relations imp. E.R. $\nu^{(1)}\int f^{(1)}(x)\psi(x)dx=\int\delta(x)\psi(x)dx$ and imp. E.R. $\nu^{(2)}\int f^{(2)}(x)\psi(x)dx=\int\delta(x)\psi(x)dx$ holds valid for any $\psi(x)\in(\mathfrak{B})$.

When these $\phi^{(1)}(x)$ and $\phi^{(2)}(x)$ are used two space-like points $(\tilde{x},\tilde{x}')$, (or two functions depending on $\tilde{x}-\tilde{x}'$ with the same properties related to δ are used), $[\text{E.R.}\,\nu^{(1)}\varphi * f^{(1)}, \text{E.R.}\,\nu^{(2)}\pi * f^{(2)}]=0$ holds valid, and the ordinary causality condition is satisfied in a sense. Here $f^{(1)}(x)$ and $f^{(2)}(x)$ are functions defined in Lemma 4.

By the similar argument we can also assert this causality condition for three dimensional E.R. ν singular cut-off.

It seems that the value of mollifier in our model represents a sort of probability of the existence of positive and negative particles (the component of elementary particle under the quantized Brownian motion). Furthermore, the use of the above $\nu^{(1)}$ and $\nu^{(2)}$ etc. seems to give a connection between local and nonlocal theory.

References

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