115. *The Characters of Irreducible Representations of the Lorentz Group of n-th Order.*

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1. The connected component of the identity element of the orthogonal group associated with the indefinite quadratic form $x_1^2+x_2^2+\cdots+x_{n-1}^2-x_n^2$ is called the Lorentz group of n-th order and denoted by L_n.

We use the same definitions and notations as in [1] and [2]. In these papers we discussed infinite dimensional algebraically irreducible representations of the Lie algebra of L_n. We can prove that there exist complete irreducible representations of the group L_n in Hilbert spaces, which correspond to the representations of the Lie algebra listed in [2].

In this note we give the explicite formulae of the characters of these irreducible representations. Representations $\mathfrak{D}_{(\alpha;c)}$ can be constructed by the method of "induced representation" and in the series of thus constructed induced representations, some exceptional ones are not irreducible and they split into irreducible representations $\mathfrak{S}_\mu$, $D^j_{(\alpha;p)}$, $D^+_{(\alpha;p)}$, or $D^-_{(\alpha;p)}$ (semi-reducible). The diagrams of these splitting are known from the infinitesimal stand point. The characters of the induced representations are calculated by integration of some integral kernels. Using the thus calculated characters of $\mathfrak{D}_{(\alpha;c)}$ and the character formulae of finite dimensional representations, we can obtain, for instance when $n=2k+3$, successively the characters of $D^k_{(\alpha;p)}, D^{k-1}_{(\alpha;p)}, \cdots, D^1_{(\alpha;p)}$ and of the direct sum of $D^+_{(\alpha;p)}$ and $D^-_{(\alpha;p)}$. It needs some additional discussions to obtain the character of each $D^+_{(\alpha;p)}$ and $D^-_{(\alpha;p)}$ separately.

Here our discussions are restricted on one-valued representations, but the analogous results can be obtained for two-valued representations by the same method.

2. First we consider the case when n is odd: $n=2k+3$ $(k=0, 1, 2, \cdots)$.

The regular elements of L_n are divided into two classes G_1 and G_2. Every element $g\in G_1$ has eigenvalues $1, e^{i\varphi_1}, e^{-i\varphi_1}, \cdots, e^{i\varphi_k}, e^{-i\varphi_k}, e^t$, and e^{-t} (three of them are real positive) and we put $\lambda_r=e^{i\varphi_r}$, $\lambda_{-r}=e^{-i\varphi_r}$ $(r=1, 2, \cdots, k)$, $\lambda_{k+1}=e^t$, and $\lambda_{-(k+1)}=e^{-t}$. For $g\in G_2$, its eigenvalues are $1, e^{i\varphi_1}, e^{-i\varphi_1}, \cdots, e^{i\varphi_{k+1}}$, and $e^{-i\varphi_{k+1}}$ (all except 1 are complex) and we put $\lambda_r=e^{i\varphi_r}$ and $\lambda_{-r}=e^{-i\varphi_r}$ ($r=1, 2, \cdots, k+1$, and $i=\sqrt{-1}$).

i) *Character $\pi(g)$ of representation* $\mathfrak{D}_{(\alpha;c)}$.

The row of integers $\alpha=(n_1, n_2, \cdots, n_k)$ satisfying the condition $0\leq n_1\leq n_2\leq\cdots\leq n_k$ represents a highest weight of an irreducible representation of the subgroup Γ_n of L_n, isomorphic to $SO(2k+1)$. Let $\chi^\alpha(\gamma)$ $(\gamma\in\Gamma_n)$ be its character. Then for $g\in G_2$, $\pi(g)=0$, and for $g\in G_1$

$$\pi(g)=\frac{(\lambda^c_{k+1}+\lambda^{-c}_{k+1})\chi^\alpha(\gamma)}{\lambda_{k+1}^{-(k+\frac{1}{2})}\cdot|\lambda_{k+1}-1|\cdot\prod_{r=1}^{k}|\lambda_{k+1}-\lambda_r|^2}, \tag{1}$$

where γ is an element of Γ_n which has eigenvalues $1, \lambda_r$, and λ_{-r} $(r=1, 2, \cdots, k)$.

Put $l_r=n_r+(r-1/2)$ $(r=1, 2, \cdots, k)$

and

$$\mathrm{D}_1(g)=|\lambda^{\frac{1}{2}}_{n+1}-\lambda^{-\frac{1}{2}}_{k+1}|\cdot\prod_{r=1}^{k}(\lambda_r^{\frac{1}{2}}-\lambda_r^{-\frac{1}{2}})\cdot\prod_{k+1\geq r>s\geq 1}\{(\lambda_r+\lambda_{-r})-(\lambda_s+\lambda_{-s})\},$$

then (1) is rewritten as follows:

$$\pi(g)=\frac{(\lambda^c_{k+1}+\lambda^{-c}_{k+1})}{\mathrm{D}_1(g)}\cdot\begin{vmatrix}\lambda_1^{l_1}-\lambda_1^{-l_1}, & \lambda_1^{l_2}-\lambda_1^{-l_2}, & \cdots, & \lambda_1^{l_k}-\lambda_1^{-l_k}\\ \lambda_2^{l_1}-\lambda_2^{-l_1}, & \lambda_2^{l_2}-\lambda_2^{-l_2}, & \cdots, & \lambda_2^{l_k}-\lambda_2^{-l_k}\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ \lambda_k^{l_1}-\lambda_k^{-l_1}, & \lambda_k^{l_2}-\lambda_k^{-l_2}, & \cdots, & \lambda_k^{l_k}-\lambda_k^{-l_k}\end{vmatrix}. \tag{1'}$$

According to H. Weyl, we donote the determinant in the numerator by $|\lambda^{l_1}-\lambda^{-l_1}, \lambda^{l_2}-\lambda^{-l_2}, \cdots, \lambda^{l_k}-\lambda^{-l_k}|_{\lambda=\lambda_1,\lambda_2,\cdots,\lambda_k}$.

ii) *Finite dimensional representation* $\mathfrak{S}_\mu$ (see [3], p. 225).

Put $l_r=n_r+(r-1/2)$ as before, then for $g\in G_1\cup G_2$

$$\pi(g)=\frac{1}{\mathrm{D}(g)}\cdot|\lambda^{l_1}-\lambda^{-l_1}, \lambda^{l_2}-\lambda^{-l_2}, \cdots, \lambda^{l_k}-\lambda^{-l_k}, \lambda^{l_{k+1}}-\lambda^{-l_{k+1}}|_{\lambda=\lambda_1,\lambda_2,\cdots,\lambda_{k+1}}, \tag{2}$$

where $\mathrm{D}(g)=\mathrm{D}_1(g)\times(\lambda^{\frac{1}{2}}_{k+1}-\lambda^{-\frac{1}{2}}_{k+1})\cdot|\lambda^{\frac{1}{2}}_{k+1}-\lambda^{-\frac{1}{2}}_{k+1}|^{-1}$.

We denote the right side of (2) by $\pi(g; l_1, l_2, \cdots, l_k, l_{k+1})$ for brevity.

iii) *Representation* $\mathrm{D}^k_{(\alpha;p)}\cdot\alpha=(n_1, n_2, \cdots, n_k)$ and

$$0\leq n_1\leq n_2\leq\cdots\leq n_{k-1}\leq p<n_k.$$

Put $l'_k=p+(k-1/2)$ and as usual $l_r=n_r+(r-1/2)$.

Then, for $g\in G_2$, $\pi(g)=(-1)\cdot\pi(g; l_1, l_2, \cdots, l'_k, l_k)$. For $g\in G_1$,

$$\pi(g)=\frac{-1}{\mathrm{D}(g)}\cdot\begin{vmatrix}\lambda^{l_1}-\lambda^{-l_1}, & \cdots, & \lambda^{l_{k-1}}-\lambda^{-l_{k-1}}, & \lambda^{l'_k}-\lambda^{-l'_k}, & \lambda^{l_k}-\lambda^{-l_k}\\ \lambda^{l_1}_{k+1}-\lambda^{-l_1}_{k+1}, & \cdots, & \lambda^{l_{k-1}}_{k+1}-\lambda^{-l_{k-1}}_{k+1}, & \lambda^{l'_k}_{k+1}-\lambda^{-l'_k}_{k+1}, & -2\lambda^{-l_k}_{k+1}\end{vmatrix}_{\lambda=\lambda_1,\lambda_2,\cdots,\lambda_k}. \tag{3}$$

Here the numerator denotes the determinant of $(k+1)$-th degree whose r-th row is $\lambda_r^{l_1}-\lambda_r^{-l_1}, \lambda_r^{l_2}-\lambda_r^{-l_2}, \cdots, \lambda_r^{l'_{k'}}-\lambda_r^{-l'_{k'}}, \lambda_r^{l_k}-\lambda_r^{-l_k}$ for $1\leq r\leq k$, and $(k+1)$-th row is $\lambda^{l_1}_{k+1}-\lambda^{-l_1}_{k+1}, \cdots, \lambda^{l'_{k'}}_{k+1}-\lambda^{-l'_{k'}}_{k+1}, \lambda^{l_k}_{k+1}-\lambda^{-l_k}_{k+1}$. In the formula

(3), we denote by λ_{k+1} the real positive eigenvalue which is larger than 1, and therefore in general case, λ_{k+1}^{-l} should be $e^{-l|t|}$ and D(g) should be $D_1(g)$.

In the following of this section $\lambda_{k+1}>1$ is always assumed for $g\in G_1$.

iv) We consider representation $D^j_{(\alpha;p)}$ ($j=1, 2, \cdots, k$) in general. Here the order of integers in a parameter (α; p) is

$$0\leq n_1\leq n_2\leq\cdots\leq n_{j-1}\leq p<n_j\leq\cdots\leq n_k,$$

and $0\leq p<n_1\leq\cdots\leq n_k$ especially for $D^1_{(\alpha;p)}$.

Put $l'_j=p+(j-1/2)$.

For $g\in G_1$

$$\pi(g)=\frac{(-1)^{k-j+1}}{D(g)} \tag{4}$$
$$\times\begin{vmatrix}\lambda^{l_1}-\lambda^{-l_1}, \cdots, \lambda^{l_{j-1}}-\lambda^{-l_{j-1}}, \lambda^{l'_j}-\lambda^{-l'_j}, \lambda^{l_j}-\lambda^{-l_j}, \cdots, \lambda^{l_k}-\lambda^{-l_k}\\ \lambda_{k+1}^{l_1}-\lambda_{k+1}^{-l_1}, \cdots, \lambda_{k+1}^{l_{j-1}}-\lambda_{k+1}^{-l_{j-1}}, \lambda_{k+1}^{l'_j}-\lambda_{k+1}^{-l'_j}, -2\lambda_{k+1}^{-l_j}, \cdots, -2\lambda_{k+1}^{-l_k}\end{vmatrix}_{\lambda=\lambda_1,\lambda_2,\cdots,\lambda_k},$$

and for $g\in G_2$, $\pi(g)=(-1)^{k-j+1}\cdot\pi(g; l_1, \cdots, l_{j-1}, l'_j, l_j, \cdots, l_k)$.

v) $D^+_{(\alpha;p)}$ *and* $D^-_{(\alpha;p)}$. Parameter (α ; p) satisfies the condition $0<p\leq n_1\leq n_2\leq\cdots\leq n_k$.[1)]

First we consider the character of the direct sum of these two representations. Putting $l_0=p-1/2$,

for $g\in G_1$

$$\pi(g)=\frac{(-1)^k\cdot 2}{D(g)}\cdot\begin{vmatrix}\lambda^{l_0}-\lambda^{-l_0}, \lambda^{l_1}-\lambda^{-l_1}, \cdots, \lambda^{l_k}-\lambda^{-l_k}\\ \lambda_{k+1}^{-l_0}, \quad \lambda_{k+1}^{-l_1}, \quad \cdots, \lambda_{k+1}^{-l_k}\end{vmatrix}_{\lambda=\lambda_1,\cdots,\lambda_k}, \tag{5}$$

and for $g\in G_2$, $\pi(g)=(-1)^{k+1}\cdot\pi(g; l_0, l_1, l_2, \cdots, l_k)$.

To give the character of each $D^+_{(\alpha;p)}$ and $D^-_{(\alpha;p)}$ separately, the order of eigenvalues of g must be taken into account. Therefore it is convenient to realize L_n in the following form, or the same thing, to establish once for all an appropriate isomorphism between L_n and the group L'_n defined by the following conditions.

$$\begin{aligned}&L'_n: g\in SL(n, C), \quad {}^tg\sigma g=\sigma, \quad \tau\bar{g}\tau=g\\ &\text{the matrix element } g_{k+2,k+2}\geq 1.\end{aligned} \tag{6}$$

Here ${}^tg$ and $\bar{g}$ are the transposed matrix and the complex conjugate matrix of g respectively,

and

$$\sigma=\sigma_n=\begin{Vmatrix} & & & 1\\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{Vmatrix}, \qquad \tau=\begin{Vmatrix} & & \sigma_{k+1}\\ & -1 & \\ \sigma_{k+1} & & \end{Vmatrix}. \tag{7}$$

1) On p. 259 of [2], the inequality $0<p<n_1$ in 4) should be $0<p\leq n_1$.

On p. 262 of the same paper, the condition $\rho=0$ must be put in i′), and the two-valued representation $\mathfrak{D}_{(\alpha;0)}$ is direct sum of $D^+_{(\alpha;1/2)}$ and $D^-_{(\alpha;1/2)}$, as for $SL(2, R)$.

Denote a diagonal matrix with diagonal elements $a_1, a_2, \cdots, a_n$ by

$$d(a_1, a_2, \cdots, a_n).$$

For $g \in G_1$, the characters of $D^+_{(\alpha;p)}$ and $D^-_{(\alpha;p)}$ are identical and equal to the half of (5). If $g \in G_2$, g is conjugate in L'_n a diagonal element $d(\lambda_{k+1}, \lambda_k, \cdots, \lambda_1, 1, \lambda_{-1}, \cdots, \lambda_{-(k+1)})$, where $\lambda_r = e^{i\varphi_r}$ and $\lambda_{-r} = e^{-i\varphi_r}$ $(r=1, 2, \cdots, k+1)$. Then the character of $D^+_{(\alpha;p)}$ is

$$\pi(g) = \frac{(-1)^{k+1}}{2D(g)} \cdot \{ | \lambda^{l_0} - \lambda^{-l_0}, \lambda^{l_1} - \lambda^{-l_1}, \cdots, \lambda^{l_k} - \lambda^{-l_k} |_{\lambda=\lambda_1,\cdots,\lambda_{k+1}} + | \lambda^{l_0} + \lambda^{-l_0}, \lambda^{l_1} + \lambda^{-l_1}, \cdots, \lambda^{l_k} + \lambda^{-l_k} |_{\lambda=\lambda_1,\cdots,\lambda_{k+1}} \}, \tag{8}$$

and the character of $D^-_{(\alpha;p)}$ is

$$\pi(g) = \frac{(-1)^{k+1}}{2D(g)} \cdot \{ | \lambda^{l_0} - \lambda^{-l_0}, \cdots, \lambda^{l_k} - \lambda^{-l_k} | - | \lambda^{l_0} + \lambda^{-l_0}, \cdots, \lambda^{l_k} + \lambda^{-l_k} | \}. \tag{8'}$$

These characters have poles on the planes

$$\varphi_r = 0 \ (r=1, 2, \cdots, k+1).$$

3. The case when n is even: $n=2k+2$ $(k=1, 2, \cdots)$.

Also, in this case, the order of the eigenvalues of g has its meaning and for the convenience, we establish an appropriate isomorphism between L_n and the group L''_n defined in the following, and identify these two groups.

$$L''_n: g \in SL(n, C), \quad {}^t g \sigma g = \sigma, \quad \tau_1 \bar{g} \tau_1 = g, \quad g_{11} - g_{1n} - g_{n1} + g_{nn} \geq 2, \tag{9}$$

where

$$\tau_1 = \begin{Vmatrix} 1 & & \\ & \sigma_{2k} & \\ & & 1 \end{Vmatrix}.$$

A regular element g has two real positive eigenvalues and $(n-2)$ complex ones, and is conjugate in L''_n a diagonal element $d(\lambda_{k+1}, \lambda_k, \cdots, \lambda_1, \lambda_{-1} \cdots \lambda_{-(k+1)})$, where $\lambda_r = (\lambda_{-r})^{-1} = e^{i\varphi_r}$ $(r=1, 2, \cdots, k)$ and $\lambda_{k+1} = (\lambda_{-(k+1)})^{-1} = e^t$.

i) *Representations* $\mathfrak{D}_{(\alpha;p)}$. The parameter $\alpha=(n_1, n_2, \cdots, n_k)$ with the condition $|n_1| \leq n_2 \leq \cdots \leq n_k$ represents a highest weight of an irreducible representation of Γ_n, isomorphic to $SO(2k)$. Denote its character by $\chi^\alpha(\gamma)$ $(\gamma \in \Gamma_n)$ and put $\check{\alpha} = (-n_1, n_2, \cdots, n_k)$ and $l_r = n_r + (r-1)$. Then

$$\pi(g) = \frac{\lambda_{k+1}^c \chi^\alpha(\gamma) + \lambda_{k+1}^{-c} \chi^{\check{\alpha}}(\gamma)}{\lambda_{k+1}^{-\frac{k}{2}} \cdot \prod_{r=1}^{k} | \lambda_{k+1} - \lambda_r |^2} \tag{10}$$

where $\gamma = d(1, \lambda_k, \cdots, \lambda_1, \lambda_{-1}, \cdots, \lambda_{-k}, 1) \in \Gamma_n$ and[2)]

2) H. Weyl gave the character formulae for $O(n)$ in [3], but a slight modication can give this formula for $SO(n)$.

$$\chi^{\alpha}(\gamma)= \tag{11}$$
$$\frac{1}{2}\cdot\frac{|\lambda^{l_1}+\lambda^{-l_1}, \lambda^{l_2}+\lambda^{-l_2}, \cdots, \lambda^{l_k}+\lambda^{-l_k}|+|\lambda^{l_1}-\lambda^{-l_1}, \lambda^{l_2}-\lambda^{-l_2}, \cdots, \lambda^{l_k}-\lambda^{-l_k}|}{|1, \lambda+\lambda^{-1}, \cdots, \lambda^{k-1}+\lambda^{-(k-1)}|_{\lambda=\lambda_1,\cdots,\lambda_k}}.$$

Therefore putting $D(g)=\prod_{k+1\geq r>s\geq 1}\{(\lambda_r+\lambda_{-r})-(\lambda_s+\lambda_{-s})\}$,

$$\pi(g)=\frac{1}{2D(g)}\cdot\{(\lambda_{k+1}^{c}+\lambda_{k+1}^{-c})\cdot|\lambda^{l_1}+\lambda^{-l_1}, \cdots, \lambda^{l_k}+\lambda^{-l_k}| + (\lambda_{k+1}^{c}-\lambda_{k+1}^{-c})\cdot|\lambda^{l_1}-\lambda^{-l_1}, \cdots, \lambda^{l_k}-\lambda^{-l_k}|_{\lambda=\lambda_1,\cdots,\lambda_k}\}. \tag{10'}$$

ii) *Finite dimensional representation* $\mathfrak{S}_{\mu}$.[2]
Here $\mu=(n_1, n_2, \cdots, n_{k+1})$ and $|n_1|\leq n_2\leq\cdots\leq n_{k+1}$.

$$\pi(g)=\frac{1}{2D(g)}\cdot\{|\lambda^{l_1}+\lambda^{-l_1}, \cdots, \lambda^{l_k}+\lambda^{-l_k}, \lambda^{l_{k+1}}+\lambda^{-l_{k+1}}| + |\lambda^{l_1}-\lambda^{-l_1}, \cdots, \lambda^{l_k}-\lambda^{-l_k}, \lambda^{l_{k+1}}-\lambda^{-l_{k+1}}|_{\lambda=\lambda_1,\cdots,\lambda_{k+1}}\}. \tag{12}$$

iii) *Representations* $D^{j}_{(\alpha;p)}$ $(j=1, 2, \cdots, k-1)$. The parameter $(\alpha\,;p)$ satisfies $|n_1|\leq n_2\leq\cdots\leq n_j\leq p<n_{j+1}\leq\cdots\leq n_k$, and especially for $D^{1}_{(\alpha;p)}$, $|n_1|\leq p<n_2\leq\cdots\leq n_k$.

Put $l'_{j+1}=p+j$ and $l_r=n_r+(r-1)$ as before, then

$$\pi(g)=\frac{(-1)^{k-j+1}}{2D(g)}\times \tag{13}$$
$$\left\{\begin{vmatrix}\lambda^{l_1}+\lambda^{-l_1}, \cdots, \lambda^{l_j}+\lambda^{-l_j}, \lambda^{l'_{j+1}}+\lambda^{-l'_{j+1}}, \lambda^{l_{j+1}}+\lambda^{-l_{j+1}}, \cdots, \lambda^{l_k}+\lambda^{-l_k}\\ \lambda_{k+1}^{l_1}+\lambda_{k+1}^{-l_1}, \cdots, \lambda_{k+1}^{l_j}+\lambda_{k+1}^{-l_j}, \lambda_{k+1}^{l'_{j+1}}+\lambda_{k+1}^{-l'_{j+1}}, \quad 0 \quad, \cdots, \quad 0\end{vmatrix} + \begin{vmatrix}\lambda^{l_1}-\lambda^{-l_1}, \cdots, \lambda^{l_j}-\lambda^{-l_j}, \lambda^{l'_{j+1}}-\lambda^{-l'_{j+1}}, \lambda^{l_{j+1}}-\lambda^{-l_{j+1}}, \cdots, \lambda^{l_k}-\lambda^{-l_k}\\ \lambda_{k+1}^{l_1}-\lambda_{k+1}^{-l_1}, \cdots, \lambda_{k+1}^{l_j}-\lambda_{k+1}^{-l_j}, \lambda_{k+1}^{l'_{j+1}}-\lambda_{k+1}^{-l'_{j+1}}, \quad 0 \quad, \cdots, \quad 0\end{vmatrix}_{\lambda=\lambda_1\cdots\lambda_k}\right\}.$$

Especially for $D^{1}_{(\alpha;p)}$, the $(k+1)$-th row has only two non-zero elements.

4. Eigenvalues of Laplace operaters. The characters calculated above are eigendistributions of every Laplace operator of L_n. We briefly mention their eigenvalues.

I: *Case of* $n=2k+3$. L_n has two Cartan subgroups which are not mutually conjugate. The one is the subgroup A_1 of L_n corresponding to the all diagonal elements in L''_n and the other A_2 corresponds to the all diagonal elements in L'_n. The radial part of a Laplace operator Δ is determined as follows by a symmetric polynomiyal of $X_1^2, X_2^2, \cdots, X_k^2$, and X_{k+1}^2 (denote it by $P(X_1, \cdots, X_{k+1})$).

At $g\in A_1$, corresponding to $d(e^t, e^{i\varphi_k}, \cdots, 1, \cdots, e^{-i\varphi_k}, e^{-t})\in L''_n$,

$$(D(g))^{-1}P\left(\frac{1}{i}\frac{\partial}{\partial\varphi_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial\varphi_k}, \frac{\partial}{\partial t}\right)\circ D(g), \tag{14}$$

and at $g\in A_2$, corresponding to $d(e^{i\varphi_{k+1}}, e^{i\varphi_k}, \cdots, e^{-i\varphi_k}, e^{-i\varphi_{k+1}})\in L'_n$,

$$(D(g))^{-1}P\left(\frac{1}{i}\frac{\partial}{\partial\varphi_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial\varphi_k}, \frac{1}{i}\frac{\partial}{\partial\varphi_{k+1}}\right)\circ D(g). \tag{14'}$$

The eigenvalues of Δ corresponding to the character of $\mathfrak{D}_{(\alpha;c)}$, $\mathfrak{S}_{\mu}$, $D^{k}_{(\alpha;p)}, \cdots, D^{j}_{(\alpha;p)}, \cdots, D^{+}_{(\alpha;p)}$, and $D^{-}_{(\alpha;p)}$ are $P(l_1, \cdots, l_k, c)$, $P(l_1, \cdots, l_k, l_{k+1})$, $P(l_1, \cdots, l_{k-1}, l'_k, l_k), \cdots, P(l_1, \cdots, l_j, l'_{j+1}, l_{j+1}, \cdots, l_k), \cdots, P(l_0, l_1, \cdots, l_k)$, and $P(l_0, l_1, \cdots, l_k)$ resp.

To $P(1/2, 3/2, \cdots, (2k+1)/2)$, there corresponds $(k+4)=(n+1/2)+2$ irreducible unitary representations, including the identity representation of L_n (i.e. one dimensional representation).

II: *Case of* $n=2k+2$. L_n has only one Cartan subgroup A, except its conjugate ones, which corresponds to the all diagonal elements in L''_n. The radial part of a Laplace operator Δ is determined by a polynomial $P(X_1, \cdots, X_k, X_{k+1})$, which is generated by a monomial $X_1X_2\cdots X_kX_{k+1}$ and symmetric polynomials of $X_1^2, X_2^2, \cdots, X_k^2$, and X_{k+1}^2.

Thus, at $g \in A$, corresponding to $d(e^t, e^{i\varphi_k}, \cdots, e^{-i\varphi_k}, e^{-t}) \in L'_n$,

$$(D(g))^{-1}P\left(\frac{1}{i}\frac{\partial}{\partial\varphi_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial\varphi_k}, \frac{\partial}{\partial t}\right)\circ D(g). \tag{15}$$

The eigenvalues of Δ corresponding to the character of $\mathfrak{D}_{(\alpha;c)}$, $\mathfrak{S}_{\mu}$, $D^{k-1}_{(\alpha;p)}, \cdots, D^{j}_{(\alpha;p)}, \cdots$, and $D^{1}_{(\alpha;p)}$ are $P(l_1, l_2, \cdots, l_k, c)$, $P(l_1, \cdots, l_k, l_{k+1})$, $P(l_1, \cdots, l_{k-1}, l'_k, l_k), \cdots, P(l_1, \cdots, l_j, l'_{j+1}, l_{j+1}, \cdots, l_k), \cdots$, and $P(l_1, l'_2, l_2, \cdots, l_k)$ resp.

To $P(0, 1, \cdots, k)$, there corresponds $(k+1)=n/2$ irreducible unitary representations, including identity representation.

References

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