152. *On a J. v. Neumann's Theorem*

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J. v. Neumann proved the following theorem in his paper "Zur Algebra der Funktionaloperatoren und Theorie der normalen Operatoren".

In Hilbert space with the weak topology the first axiom of countability of Hausdorff does not hold.

We will give an alternative proof for this theorem.

Lemma. If $\mathfrak{B}$ is a vector basis (Hamel basis) of a Hilbert space $\mathfrak{H}$, then the cardinal number of $\mathfrak{B}$ is greater than $\aleph_0$.

Suppose that the cardinal number of $\mathfrak{B}$ is $\aleph_0$, then all elements of $\mathfrak{B}$ can be written in the following way

$$\mathfrak{B}=\{\varphi_1, \varphi_2, \cdots, \varphi_n, \cdots\}.$$

Let us consider the sets F_n $(n=1, 2, 3, \cdots)$ consisting of all linear combinations of $\varphi_1, \varphi_2, \cdots, \varphi_n$, then every set F_n is nowhere dense. Let $\mathfrak{L}(F_n)$ be the linear hull of set F_n, then, since $UFn \supseteq \mathfrak{B}$,

$$\therefore \quad \mathfrak{L}(UF_n) \supseteq \mathfrak{L}(\mathfrak{B}),$$

on the other hand $\mathfrak{L}(UF_n) = UF_n$, $\mathfrak{L}(\mathfrak{B})=\mathfrak{H}$,

$$\therefore \quad \mathfrak{H}=UF_n.$$

This contradicts the fact that in Hilbert space the Baire category theorem holds.

Proof of Theorem. Let $\mathfrak{B}$ be the vector basis (Hamel basis) of the Hilbert space $\mathfrak{H}$, let us consider the following family of sets as a system of neighborhoods for any $f_0 \in \mathfrak{H}$, and denote it by $\mathfrak{V}_1(f_0)$:

$$\mathfrak{V}_1(f_0)=\left\{u_2\left(f_0; \psi_1, \psi_2, \cdots, \psi_t, \frac{1}{n}\right) \right.$$
$$\left. \psi_1, \psi_2, \cdots, \psi_t \in \mathfrak{B}, n=1, 2, \cdots, t=1, 2, \cdots\right\},$$

where the sets $u_2\left(f_0; \psi_1, \cdots, \psi_t, \frac{1}{n}\right)$ consist of all elements $f \in \mathfrak{H}$ such that

$$|(f-f_0, \psi_1)|<\frac{1}{n}, |(f-f_0, \psi_2)|<\frac{1}{n}, \cdots$$
$$\cdots, |(f-f_0, \psi_t)|<\frac{1}{n}.$$

In Hilbert space $\mathfrak{H}$ with the weak topology the system $\mathfrak{V}(f_0)$ of neighborhoods for any element $f_0 \in \mathfrak{H}$ consists of all sets $u_2(f_0; \varphi_1, \cdots, \varphi_s, \varepsilon)$, where $\varphi_1, \varphi_2, \cdots, \varphi_s$ belong to $\mathfrak{H}$, $s=1, 2, \cdots$ and $0<\varepsilon<\infty$.

Then $\mathfrak{V}(f_0)$ is equivalent to $\mathfrak{V}_1(f_0)$.

Indeed, since it is clear that $\mathfrak{V}(f_0)\supset\mathfrak{V}_1(f_0)$, it suffices to show that for any $u\in\mathfrak{V}(f_0)$ one can find a $v\in\mathfrak{V}_1(f_0)$ such that $v\subseteq u$. Let u be $u_2(f_0;\varphi_1,\cdots,\varphi_s,\varepsilon)$. Since $\mathfrak{B}$ is the vector basis in $\mathfrak{H}$, the φ_i can be represented in the following way

$$\varphi_1=a_{11}\psi_{11}+a_{12}\psi_{12}+\cdots+a_{1n_1}\psi_{1n_1},$$
$$\varphi_2=a_{21}\psi_{21}+a_{22}\psi_{22}+\cdots+a_{2n_2}\psi_{2n_2},$$
$$\cdots\cdots\cdots\cdots$$
$$\varphi_s=a_{s1}\psi_{s1}+a_{s2}\psi_{s2}+\cdots+a_{sn_s}\psi_{sn_s}.$$

Put

$$M=\text{Max}\,\{|a_{11}|,\cdots,|a_{1n_1}|,\cdots,|a_{sn_s}|\}$$
$$N=\text{Max}\,\{n_1,n_2,\cdots,n_s\}.$$

Then let us consider the element $v\in\mathfrak{V}_1(f_0)$ such that

$$v=u_2\left(f_0;\psi_{11},\cdots,\psi_{1n_1},\psi_{21},\cdots,\psi_{sn_s},\frac{1}{n}\right),$$

where $\dfrac{1}{n}<\dfrac{\varepsilon}{MN}$.

Since, for any $f\in v$, the inequalities

$$|(f-f_0,\psi_{ij})|<\frac{1}{n},$$

holds,

$$|(f-f_0,\varphi_i)|<\varepsilon,$$
$$\therefore\quad f\in u,$$
$$\therefore\quad v\subseteq u.$$

Therefore $\mathfrak{V}_1(f_0)$ is equivalent to $\mathfrak{V}(f_0)$. We will denote this relation by $\mathfrak{V}(f_0)\sim\mathfrak{V}_1(f_0)$. (1)

Now we will prove that in Hilbert space with the weak topology the first axiom of countability does not hold.

Assume that the first axiom of countability of Hausdorff holds in the Hilbert space $\mathfrak{H}$ with the weak topology, i.e., for any $f_0\in\mathfrak{H}$, there exists a countable system $\mathfrak{V}_0(f_0)$ of neighborhoods which is equivalent to $\mathfrak{V}(f_0)$. From (1)

$$\mathfrak{V}_0(f_0)\sim\mathfrak{V}_1(f_0). \qquad (2)$$

By this relation, for any $u=u_2(f_0;\varphi_1,\cdots,\varphi_s,\varepsilon)\in\mathfrak{V}_0(f_0)$, there exists a $v=u_2\left(f_0;\psi_1,\cdots,\psi_t,\dfrac{1}{n}\right)\in\mathfrak{V}_1(f_0)$ such that $v\subseteq u$. Let $\mathfrak{M}$ be the set of all linear combinations of $\psi_1,\cdots,\psi_t$ which belong to $\mathfrak{B}$ and let $\mathfrak{M}'$ be the orthogonal complement of $\mathfrak{M}$. Then for any $f\in\mathfrak{M}'$, we have

$$(f,\psi_i)=0,\qquad i=1,2,\cdots,t,$$

and therefore, for an arbitrary number λ,

$$\lambda f+f_0\in v$$
$$\therefore\quad \lambda f+f_0\in u$$
$$\therefore\quad |\lambda||(f,\varphi_i)|<\varepsilon.$$

Since λ is an arbitrary number,

$$(f, \varphi_i)=0, \qquad i=1, 2, \cdots, s.$$

Hence

$$\varphi_i \in (\mathfrak{M}')'=\mathfrak{M}.$$

Thus the φ_i are linear combinations of $\psi_1, \psi_2, \cdots, \psi_t$. Analogously, if $u=u_2(f_0; \varphi_1, \cdots, \varphi_s, \varepsilon)\subseteq v=u_2\left(f_0; \psi_1, \cdots, \psi_t, \frac{1}{n}\right)$, then the ψ_i are linear combinations of $\varphi_1, \varphi_2, \cdots, \varphi_s$.

By the assumption that $\mathfrak{B}_0(f_0)$ is a countable system, the elements of $\mathfrak{B}_0(f_0)$ can be written in the following way

$$\mathfrak{B}_0(f_0)=\{u_1, u_2, \cdots, u_n, \cdots\}$$

where $u_i=u_2(f_0; \varphi_{i1}, \varphi_{i2}, \cdots, \varphi_{is_i}, \varepsilon_i)$, $i=1, 2, \cdots$. Since $\mathfrak{B}_0(f_0)\sim\mathfrak{B}_1(f_0)$, for any $v=u_2\left(f_0; \psi_1, \cdots, \psi_t, \frac{1}{n}\right)\in\mathfrak{B}_1(f_0)$, it immediately follows that one can find an element $u_i=u_2(f_0; \varphi_{i1}, \cdots, \varphi_{is_i}, \varepsilon_i)\in\mathfrak{B}_0(f_0)$ for which

$$u_i\subseteq v. \tag{3}$$

By what has been said above the ψ_i are linear combinations of elements $\varphi_{i1}, \cdots, \varphi_{is_i}$, and furthermore, the φ_{ik} are also linear combinations of elements $\psi_1', \psi_2', \cdots, \psi_l'$ which belong to $\mathfrak{B}$. Since the elements belonging to $\mathfrak{B}$ are linearly independent, the number of elements $v=u_2\left(f_0; \psi_1, \cdots, \psi_t, \frac{1}{n}\right)\in\mathfrak{B}$ which satisfy the relation (3) is finite. Therefore the cardinal number of elements in $\mathfrak{B}$ is $\aleph_0$, which contradicts the Lemma. Thus the proof is completed.

References

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