145. On Axiom Systems of Propositional Calculi. VII

By Yoshinari ARAI and Kiyoshi ISÉKI

(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1965)

In his paper [1], Y. Arai, one of the present authors, obtained deductions of several axiom systems for propositional calculus from the (L_3)-axioms:

1 $CpCqp$,

2 $CCpCqrCCpqCpr$,

3 $CCNpNqCqp$.

In this note, we shall show the other deductions from the (L_3)-axioms and give some remarks. For the deductions, we use rules of substitution and detachment as in our previous notes.

The first two axioms imply the following theses:

1 $p/CCpCqrCCpqCpr$, q/Cqr $*C2$—4,

4 $CCqrCCpCqrCCpqCpr$.

2 p/Cqr, $q/CpCqr$, $r/CCpqCpr$, $*C4$—$C1$ p/Cqr, q/p—5,

5 $CCqrCCpqCpr$.

1 $p/CpCpq$, $q/CCpqCpr$ $*C1$ p/q, q/p—6,

6 $CCCpqCprCqCpq$.

5 p/q, q/Cpq, r/Cpr—7,

7 $CCCpqCprCCqCpqCqCpr$.

2 $p/CCpqCpr$, $q/CqCpq$, $r/CqCpr$ $*C7$—$C6$—8,

8 $CCCpqCprCqCpr$.

5 $p/CpCqr$, $q/CCpqCpr$, $r/CqCpr$ $*C8$—$C2$—9,

9 $CCpCqrCqCpr$.

9 p/Cqr, q/Cpq, r/Cpr $*C5$—10,

10 $CCpqCCqrCpr$.

2 q/p, r/q—11,

11 $CCpCpqCCppCpq$.

2 q/Cqp, r/p $*C1$ q/Cqp—$C1$—12,

12 Cpp.

9 $p/CpCpq$, q/Cpp, r/Cpq $*C11$—$C12$—13,

13 $CCpCpqCpq$.

In the deductions from axioms 1 and 2, theses 5, 9, 10, and 13 are fundamental and important. To obtain thesis 10, we shall show another deduction by Y. Arai.

2 p/Cqr, q/Cpq, r/Cpr $*C5$—14,

14 $CCCqrCpqCCqrCpr$.

1 $p/CCCqrCpqCCqrCpr$, q/Cpq *C14—15,

15 $CCpqCCCqrCpqCCqrCpr$.

2 p/Cpq, $q/CCqrCpq$, $r/CCqrCpr$ *C15—C1 p/Cpq, q/Cqr—10,

10 $CCpqCCqrCpr$.

Next we shall use axiom 3 and deduce some theses.

1 $p/CCNqNpCpq$, q/Np *C3 p/q, q/p—16,

16 $CNpCCNqNpCpq$.

2 p/Np, $q/CNqNp$, r/Cpq *C16—C1 p/Np, q/Nq—17,

17 $CNpCpq$.

9 p/Np, q/p, r/q *C17—18,

18 $CpCNpq$.

5 p/NNp, $q/CNpNNNp$, $r/CNNpp$ *C3 q/NNp—C17 p/Np, $q/NNNp$—19,

19 $CNNpCNNpp$.

13 p/NNp, q/p *C19—20,

20 $CNNpp$.

3 p/NNp, q/p *C20 p/Np—21,

21 $CpNNp$.

10 p/NNp, q/p, r/q *20—22,

22 $CCpqCNNpq$.

5 p/NNp, r/NNq *C21 p/q—23,

23 $CCNNpqCNNpNNq$.

10 p/Cpq, $q/CNNpq$, $r/CNNpNNq$ *C22—C23—24,

24 $CCpqCNNpNNq$.

10 p/Cpq, $q/CNNpNNq$, $r/CNqNp$ *C24—C3 p/Np, q/Nq—25,

25 $CCpqCNqNp$.

Therefore we have Frege axioms:

1 $CpCqp$,

2 $CCpCqrCpqCpr$,

3 $CCpqCNqNp$,

4 $CNNpp$,

5 $CpNNp$.

A deduction from Frege axioms is given by S. Tanaka. The detail will be given in his later paper. Here we shall only give the proof of (F) $\Rightarrow$ (L_3). Then theses 1$\sim$14 also hold in the (F)-system. We shall freely use some of these results in the proof. In our proof, we need the followings:

6 $CCpqCCqrCpr$.

7 $CCqrCCpqCpr$.

3 q/Nq—8,

8 $CCpNqCNNqNp$.

6 p/q, q/NNq, r/Np *$C5$—9,

9 $CCNNqNpCqNp$.

6 $p/CpNq$, $q/CNNqNp$, $r/CqNp$ *$C8$—$C9$—10,

10 $CCpNqCqNp$.

7 p/q, q/NNp, r/p *$C4$—11,

11 $CCqNNpCqp$.

6 $p/CNpNq$, $q/CqNNp$, r/Cqp *$C10$ p/Nq—$C11$—12,

12 $CCNpNqCqp$.

Reference

[1] Y. Arai: On axiom systems of propositional calculi. III. Proc. Japan Acad., **41**, 570-574 (1965).

Corrections to Yasuyuki Imai and Kiyoshi Iséki: "On Axiom Systems of Propositional Calculi. I" (Proc. Japan Acad., **41**, 436-439 (1965)).

Page 438, line 36: For "10" read "6".
〃 , 〃 : For "$C2$" read "$C6$".
〃 , line 38: For "$C14$" read "$C3$".
Page 439, line 17: For "$C6$" read "$C15$".