139. *A Perturbation Theorem for Semi-groups of Linear Operators*

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Let A be the infinitesimal generator of a contraction semi-group T_t of class (C_0) on the Banach space X.[1] A is thus a closed linear operator with the domain $D(A)$ and the range $R(A)$ both in X such that: i) $D(A)$ is dense in X, and ii) the resolvent $(\lambda I-A)^{-1}$ exists as a bounded linear operator on X into X satisfying the estimate $\|\lambda(\lambda I-A)^{-1}\|\leqq 1$ for all $\lambda>0$. Let B likewise be the infinitesimal generator of another contraction semi-group of class (C_0) on X. Then the condition $D(B)\geqq D(A)$ implies, by the closed graph theorem, that there exist positive constants a and b such that

$$\|Bx\|\leqq a\|Ax\|+b\|x\| \quad \text{for} \quad x\in D(A).$$

As an important remark to Theorem 2 in H. F. Trotter [1] (Cf. T. Kato [1]), E. Nelson [1] proved that $(A+B)$ with the domain $D(A+B)=D(A)$ is the infinitesimal generator of a contraction semi-group of class (C_0) if we can take $a<1/2$.

The purpose of the present note is to propose a sufficient condition in order that Nelson's hypothesis be satisfied. We shall prove

Theorem. Let $0<\alpha<1$. Let $\hat{A}_\alpha=-(-A)^\alpha$ be the fractional power of A, and let us assume that $D(B)\geqq D(\hat{A}_\alpha)$. Then $(A+B)$ with the domain $D(A+B)=D(A)$ is the infinitesimal generator of a contraction semi-group of class (C_0) on X.

Corollary. Assume, furthermore, that A generates a holomorphic semi-group, then $(A+B)$ with the domain $D(A+B)=D(A)$ also generates a holomorphic semi-group.

Remark 1.[2] The fractional power $\hat{A}_\alpha$ is defined as the infinitesimal generator of the semi-group of class (C_0):

$$(1)\qquad \hat{T}_{t,\alpha}x=\int_0^\infty f_{t,\alpha}(s)T_s x\,ds \quad (t>0,\ x\in X),$$

where

$$(2)\qquad f_{t,\alpha}(s)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{zs-tz^\alpha}dz(\sigma>0,\ t>0,\ s\geqq 0),$$

the branch of z^α being taken so that $Re(z^\alpha)>0$ for $Re(z)>0$. According to V. Balakrishnan [1], we have $D(\hat{A}_\alpha)\geqq D(A)$ and

$$(3)\qquad (-A)^\alpha x=\frac{\sin\pi\alpha}{\pi}\int_0^\infty \lambda^{\alpha-1}(\lambda I-A)^{-1}(-Ax)d\lambda \quad \text{for } x\in D(A).$$

1) See, e.g., E, Hille-R. S. Phillips [1] or K. Yosida [1].
2) See, e.g., K. Yosida [1].

Remark 2.[3] A contraction semi-group T_t of class (C_0) is called a holomorphic semi-group if there exist positive numbers C_1 and C_2 $(<C_1)$ such that T_t admits holomorphic extension T_λ given by Taylor's expansion

(4) $$T_\lambda x=\sum_{n=0}^{\infty}(\lambda-t)^n T_t^{(n)}x/n! \text{ for } x\in X \text{ and } |\arg\lambda|<\tan^{-1}C_1$$

satisfying the estimate

(5) $$\|e^{-\lambda}T_\lambda\|\leqq C_2 \text{ for } |\arg\lambda|<\tan^{-1}C_2,$$

the existence of the strong derivatives $T_t^{(n)}x$ with respect to t being assumed for all $x\in X$. It is known that a contraction semi-group T_t of class (C_0) is a holomorphic semi-group if and only if, for fixed $\sigma_0>0$, we have the estimate

(6) $$\overline{\lim_{|\tau|\uparrow\infty}}\|\tau((\sigma_0+i\tau)I-A)^{-1}\|<\infty.$$

Proof of the Theorem. Being infinitesimal generators of contraction semi-groups of class (C_0), A and B are both dissipative in the sense of G. Lumer and R. S. Phillips [1]. That is, if we take, for $x\in X$, a continuous linear functional $\varphi=\varphi_x$ defined on X such that

$$\|\varphi_x\|=1 \text{ and } \langle x,\varphi_x\rangle=\|x\|,$$

then we have

$$Re\langle Ax,\varphi_x\rangle\leqq 0\ (x\in D(A)) \text{ and } Re\langle Bx,\varphi_x\rangle\leqq 0\ (x\in D(B)).$$

Thus $Re\langle(\lambda I-A-B)x,\varphi_x\rangle\geqq\lambda\|x\|$ and hence $\|(\lambda I-A-B)x\|\geqq\lambda\|x\|$ for all $x\in D(A)$ and $\lambda>0$. Therefore

(7) the inverse $(\lambda I-A-B)^{-1}$ exists with the estimate

$$\|(\lambda I-A-B)^{-1}f\|\leqq\lambda^{-1}\|f\| \text{ for all } f\in R(\lambda I-A-B).$$

Since $D(A+B)=D(A)$ is dense in X, we have only to prove that $R(\lambda I-A-B)=X$ for some $\lambda>0$. For, then it is easy to show that $(\lambda I-A-B)^{-1}$ is everywhere defined on X with the estimate

$$\|\lambda(\lambda\tau-A-B)^{-1}\|\leqq 1 \text{ for all } \lambda>0.$$

Now formula (3) may be written as

$$(-A)^\alpha x=\frac{\sin\pi\alpha}{\pi}\left\{\int_0^\delta\lambda^{\alpha-1}[I-\lambda(\lambda I-A)^{-1}]x\,d\lambda+\int_\delta^\infty\lambda^{\alpha-2}\lambda(\lambda I-A)^{-1}(-Ax)d\lambda\right\}.$$

Hence, by ii), we obtain

(8) $$\|\hat{A}_\alpha x\|\leqq a\|Ax\|+b\|x\| \text{ for all } x\in D(A),$$

where a can be taken arbitrarily small by taking b appropriately.[4]

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3) See, e.g., K. Yosida [1].

4) Mr. K. Masuda kindly called the author's attention that (8) may be replaced by a sharper one:

(8)′ $$\|\hat{A}_\alpha x\|\leqq\frac{\sin\pi\alpha}{\pi}\frac{2^{1-\alpha}}{\alpha(1-\alpha)}\|Ax\|^\alpha\|x\|^{1-\alpha} \text{ for all } x\in D(A),$$

which is obtained by considering the minimum with respect to δ of

$$\frac{\sin\pi\alpha}{\pi}\left\{\|x\|\int_0^\delta 2\cdot\lambda^{\alpha-1}d\lambda+\|Ax\|\int_\delta^\infty\lambda^{\alpha-2}d\lambda\right\}.$$

We also have, by $D(B)\geqq D(\hat{A}_\alpha)$ and the closed graph theorem, that there exists a positive constant c such that

(9) $$\| Bx \| \leqq c(\| \hat{A}_\alpha x \| + \| x \|) \text{ for all } x \in D(\hat{A}_\alpha).$$

Thus, by (8) and (9),

(10) $$\| Bx \| \leqq ac \| Ax \| + c(b+1) \| x \| \text{ for all } x \in D(A).$$

Since $D(A)\leqq D(\hat{A}_\alpha)\leqq D(B)$, the operator $B(\lambda I-A)^{-1}$ with $\lambda>0$ is everywhere defined on X and so, by (10) and the closed graph theorem, we obtain

(11) $$\| B(\lambda I-A)^{-1} \| \leqq ac \| A(\lambda I-A)^{-1} \| + c(b+1) \| (\lambda I-A)^{-1} \|.$$

Hence, by ii) and $A(\lambda I-A)^{-1}=\lambda(\lambda I-A)^{-1}-I$, we see that $\|B(\lambda I-A)^{-1}\|<1$ for $2ac+\lambda^{-1}c(b+1)<1$. This proves that, for $2ac<1$ and for sufficiently large $\lambda>0$, the inverse $(I-B(\lambda I-A)^{-1})^{-1}=\sum_{m=0}^{\infty}(B(\lambda I-A)^{-1})^m$ exists as a bounded linear operator on X into X so that $R((I-B(\lambda I-A)^{-1})=X$. Therefore, by $R(\lambda I-A)=X$, we see that $R(\lambda I-A-B)=R((I-B(\lambda I-A)^{-1})(\lambda I-A))=X$.

Proof of the Corollary. Since A and $(A+B)$ both generate contraction semi-groups of class (C_0), we know that $((\sigma_0+i\tau)I-A-B)^{-1}$ and $((\sigma_0+i\tau)I-A)^{-1}$ both exist as bounded linear operators on X into X with the estimates

$$\| \sigma_0((\sigma_0+i\tau)I-A-B)^{-1} \| \leqq 1, \quad \| \sigma_0((\sigma_0+i\tau)I-A)^{-1} \| \leqq 1$$

and $$\lim_{|\tau|\uparrow\infty} |\tau| \, \| ((\sigma_0+i\tau)I-A)^{-1} \| < \infty.$$

Hence, as in the proof of Theorem, we prove that for $2ac<1$ and for sufficiently large $|\tau|$, the estimate

$$\| \tau((\sigma_0+i\tau)I-A-B)^{-1} \| \leqq \| (I-B((\sigma_0+i\tau)I-A)^{-1})^{-1} \| \cdot \| \tau((\sigma_0+i\tau)I-A)^{-1} \|$$

so that $|\tau| \cdot \| ((\sigma_0+i\tau)I-A-B)^{-1} \|$ is bounded as $|\tau|\uparrow\infty$. This proves that $(A+B)$ generates a holomorphic semi-group.

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