167. Monotone Sequence of 0-dimensional Subsets of Metric Spaces

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Let ω_1 be the first uncountable ordinal and $\omega(\mathfrak{c})$ the first ordinal whose power is $\mathfrak{c}$. This paper proves the following two theorems.

Theorem 1. *Let X be a metric space which is the countable sum of 0-dimensional subsets. Then there exists a sequence $\{J_i : i<\omega\}$ of subsets of X such that* i) $J_i \subset J_{i+1}$ *for every* i, ii) $\dim J_i \leqq 0$ *for every* i, *and* iii) $\bigcup J_i = X$.

Theorem 2. *Let X be a non-empty metric space. Then there exists a transfinite sequence $\{J_\alpha : \alpha<\omega_1\}$ of subsets of X such that* i) $J_\alpha \subset J_\beta$ *whenever* $\alpha<\beta$, ii) $\dim J_\alpha \leqq 0$ *for every* α, *and* iii) $\bigcup J_\alpha = X$.

In both cases we use the following notations, where ρ is the preassigned metric on X. Take two sequences $\mathfrak{U}_{ij}=\{U_\lambda : \lambda \in A_{ij}\}$ and $\mathfrak{F}_{ij}=\{F_\lambda : \lambda \in A_{ij}\}$, where $i, j=1, 2, \cdots$, which satisfy the following conditions (cf. Bing [1]):

(1) $\mathfrak{U}_{ij}$ is a discrete collection of open sets of X.

(2) $\mathfrak{F}_{ij}$ is a collection of non-empty closed sets of X.

(3) $F_\lambda \subset U_\lambda$ for every $\lambda \in A$, where $A=\bigcup_{i\,j} A_{ij}$.

(4) $\mathfrak{F}_i=\{F_\lambda : \lambda \in A_i\}$ covers X for every i, where $A_i=\bigcup_j A_{ij}$.

(5) $\mathfrak{U}_i=\{U_\lambda : \lambda \in A_i\}$ is locally finite.

(6) $\rho(\mathfrak{U}_i)<1/i$.

Set $U_{ij}=\bigcup\{U_\lambda : \lambda \in A_{ij}\}$ and $F_{ij}=\bigcup\{F_\lambda : \lambda \in A_{ij}\}$.

Proof of Theorem 1. Let I' be the set of all rational numbers r with $0<r<1$. By Nagata [4, Lemma 4.1] there exists a collection $\{U_{ijr} : i, j=1, 2, \cdots, r \in I'\}$ of open sets of X which satisfies the following conditions:

(7) $F_{ij} \subset U_{ijr} \subset \bar{U}_{ijr} \subset U_{ijs} \subset \bar{U}_{ijs} \subset U_{ij}$ for $r<s$.

(8) $\{B(U_{ijr})=\bar{U}_{ijr}-U_{ijr} : i, j=1, 2, \cdots, r \in I'\}$ is pointfinite.

Let $I'=\{r_1, r_2, \cdots\}$ and $J_i=X-\bigcup\{B(U_{jkr}) : j, k=1, 2, \cdots, r \in I'-\{r_1, \cdots, r_i\}\}$. Then by Morita [3, Lemma 3.3] $\dim J_i \leqq 0$. It is evident that $J_1 \subset J_2 \subset \cdots$. To see $\bigcup J_i = X$ let x be an arbitrary point of X. By (8) there exists i such that $x \notin B(U_{ijr})$ for any i, j and any $r \in I'-\{r_1, \cdots, r_i\}$. Hence $x \in J_i$ and the proof is completed.

Proof of Theorem 2. Let I be the unit interval $[0, 1]$ and $\{I_\alpha : \alpha<\omega(\mathfrak{c})\}$ the family of all residue classes of I *modulo* the rational numbers. Set

$$L_\alpha=\bigcup\{I_\beta : \beta \leqq \alpha \text{ or } \omega_1 \leqq \beta<\omega(\mathfrak{c})\},\ \alpha<\omega_1.$$

Then we have a sequence $\{L_\alpha : \alpha<\omega_1\}$ such that i) $L_\alpha \subset L_\beta$ whenever $\alpha<\beta$, ii) $\dim L_\alpha=0$ for every α, and iii) $\bigcup L_\alpha=I$. This sequence is taken from Dowker [2].

Let $f_{ij}: X\to I$ be a continuous function such that i) $f_{ij}(x)=0$ if $x\in X-U_{ij}$ and ii) $f_{ij}(x)=1$ if $x\in F_{ij}$. Put

$$\sigma(x, y)=\rho(x, y)+\sum_{i,j}(|f_{ij}(x)-f_{ij}(y)|)/2^{i+j}.$$

Then σ is an equivalent metric to ρ such that $\sigma(F_{ij}, X-U_{ij})=d_{ij}>0$ for every i, j. For any t with $0<t\leqq 1$ set

$$H'(i, j, t)=\{x: \sigma(x, F_{ij})=t\},$$
$$H(i, j, t)=H'(i, j, t)\cap U_{ij},$$
$$J_\alpha=X-\bigcup\{H(i, j, t): i, j=1, 2, \cdots, 0<t\in I-L_\alpha\}.$$

Then $\{J_\alpha : \alpha<\omega_1\}$ satisfies the required conditions. The inequalities $J_0\subset J_1\subset \cdots J_\alpha\subset \cdots$ comes from the fact that $L_0\subset L_1\subset \cdots L_\alpha\subset \cdots$.

Let us prove $\dim J_\alpha\leqq 0$. Since $I-L_\alpha$ is dense in I, we can pick a number t_{ij} from $I-L_\alpha$ with $0<t_{ij}<d_{ij}$ for every i and j. Set

$$V_{ij}=\{x: \sigma(x, F_{ij})<t_{ij}\},$$
$$\mathfrak{V}_{ij}=\{V_\lambda=U_\lambda\cap V_{ij} : \lambda\in A_{ij}\}.$$

By (4) and (5) $\mathfrak{V}_i=\{V_\lambda : \lambda\in A_i\}$ is a locally finite open covering of X. By (6) and by the fact that $\mathfrak{U}_i$ refines $\mathfrak{U}_i$ the mesh of $\mathfrak{V}_i$ with respect to ρ is less than $1/i$. Hence $\{V_\lambda : \lambda\in A\}$ is a σ-locally finite open base of X. Let λ be an arbitrary index from A_{ij}. Since $\bar{V}_\lambda-V_\lambda\subset \bar{V}_{ij}-V_{ij}\subset H'(i, j, t_{ij})\subset U_{ij}$, $\bar{V}_\lambda-V_\lambda$ does not meet J_α. By Morita [3, Lemma 3.3] we get $\dim J_\alpha\leqq 0$.

To prove $\bigcup J_\alpha=X$ let x be an arbitrary point of X. Set

$$L=\{h(i, j)=\sigma(x, F_{ij}): 0<h(i, j)\leqq 1,\ x\in U_{ij}-F_{ij}\}.$$

Since L is countable, there exists $\beta<\omega_1$ with $L\subset L_\beta$. If $0<t\in I-L_\beta$ and $h(i, j)\in L$, then $x\notin H(i, j, t)$. If either $x\in X-U_{ij}$, $x\in F_{ij}$ or $h(i, j)>1$, then $x\notin H(i, j, t)$ for any t. Therefore $x\notin H(i, j, t)$ for any i, j, and t with $0<t\in I-L_\beta$, which implies $x\in J_\beta$. The proof is completed.

References

[1] R. H. Bing: Metrization of topological spaces. Canad. J. Math., **3**, 175-186 (1951).

[2] C. H. Dowker: Local dimension of normal spaces. Quart. J. Math. Oxford (2), **6**, 101-120 (1955).

[3] K. Morita: Normal families and dimension theory for metric spaces. Math. Ann., **128**, 350-362 (1954).

[4] J. Nagata: On the countable sum of zero-dimensional metric spaces. Fund. Math., **48**, 1-14 (1960).