200. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. XVIII

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Theorem 48. Let $\chi(\lambda)$ and σ be the same notations as before; and let $\hat{T}(\rho)$ denote the definite integral $\frac{1}{2\pi}\int_0^{2\pi}\log\sqrt{1+|\chi(\rho e^{-it})|^2}\,dt$. Then $\hat{T}(\rho)$ is not only a monotone decreasing function of ρ but also a convex function of $\log\rho$ in the interval $\sigma<\rho<\infty$.

Proof. Let c be any finite value (inclusive of zero); let $\hat{n}(\rho, c)$ denote the number of c-points, with due count of multiplicity, of $\chi(\lambda)$ in the domain $D_\rho\{\lambda: \rho<|\lambda|\leqq\infty\}$ with $\sigma<\rho<\infty$; let $\hat{n}(\infty, c)$ denote the number of c-points of $\chi(\lambda)$ at $\lambda=\infty$, that is, let $\hat{n}(\infty, c)$ be k or 0 according as c is 0 or not, on the assumption that the point at infinity is a zero-point of order k of $\chi(\lambda)$; let $C_{-\hat{n}(\infty,c)}$ denote C_{-k} or 1 according as c is 0 or not; and let

$$\hat{N}(\rho, c)=\int_\rho^\infty \frac{\hat{n}(r, c)-\hat{n}(\infty, c)}{r}dr-\hat{n}(\infty, c)\log\rho \qquad (\sigma<\rho<\infty),$$

$$\hat{m}(\rho, c)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{1}{[\chi(\rho e^{-it}), c]}dt \qquad (\sigma<\rho<\infty),$$

and

$$P(c)=\log|C_{-\hat{n}(\infty,c)}|-\left[1-\frac{\hat{n}(\infty, c)}{k}\right]\log\sqrt{1+\frac{1}{|c|^2}},$$

where we may and do assume that $\left[1-\frac{\hat{n}(\infty, c)}{k}\right]\log\sqrt{1+\frac{1}{|c|^2}}$ vanishes at $c=0$. Then

$$\hat{N}(\rho, c)+\hat{m}(\rho, c)+P(c)=\begin{cases}N(\rho, c)+m(\rho, c)-m(\infty, c) & (c\neq 0)\\ \tilde{N}(\rho, 0)+\tilde{m}(\rho, 0) & (c=0),\end{cases}$$

where $N(\rho, c)$, $\tilde{N}(\rho, c)$, $m(\rho, c)$, $\tilde{m}(\rho, 0)$, and $m(\infty, c)$ are the same notations as those used in Theorems 46 and 47. Let A be the Riemann sphere, a sphere with unit diameter touching the complex λ-plane at the origin 0, and $d\omega(c)$ an areal element at a unique point on A corresponding to a point c in that λ-plane. Since, as can be found from the geometrical meaning of $[\chi(\rho e^{-it}), c]$,

$$\iint_A \log\frac{1}{[\chi(\rho e^{-it}), c]}d\omega(c)\equiv Q$$

is a positive constant irrespective of ρe^{-it} and χ, it is obvious from

Theorems 46 and 47 that

$$\hat{T}(\rho)=\frac{1}{\pi}\iint_{A}\hat{N}(\rho, c)d\omega(c)+Q+\frac{1}{\pi}\iint_{A}P(c)d\omega(c),$$

where the last definite integral in the right-hand side is finite because of the fact that

$$\iint_{A}\log\sqrt{1+\frac{1}{|c|^2}}\,d\omega(c)=\int_0^{2\pi}\int_0^{\infty}\log\sqrt{1+\frac{1}{r^2}}\cdot\frac{r}{(1+r^2)^2}drd\theta\ (c=re^{i\theta})$$
$$<2\pi\int_0^{\infty}\frac{dr}{(1+r^2)^2}=\frac{\pi^2}{2}.$$

Putting $S(\rho)=\frac{1}{\pi}\iint_{A}\hat{n}(\rho, c)d\omega(c)$, we find therefore by direct computation that $\hat{T}'(\rho)=-\frac{S(\rho)}{\rho}<0$ for $\sigma<\rho<\infty$. Since, moreover, $\hat{T}(\rho)\to 0$ $(\rho\to\infty)$, $\hat{T}(\rho)=\int_{\rho}^{\infty}\frac{S(\rho)}{\rho}d\rho$ and $\frac{d^2\hat{T}(\rho)}{d(\log\rho)^2}=-S'(\rho)\rho$. Here $S(\rho)$ is a monotone decreasing function of ρ in the interval $\sigma<\rho<\infty$ according to the definition of $\hat{n}(\rho, c)$, so that $\frac{d^2\hat{T}(\rho)}{d(\log\rho)^2}\geqq 0$ in that interval.

With these results, the proof of the theorem is complete.

Theorem 49. Let $T(\lambda)$, $\chi(\lambda)$, and σ be the same notations as those stated in Theorem 46; let the ordinary part $R(\lambda)$ of $T(\lambda)$ be a finite constant ξ, not 0; let c denote any finite complex number different from ξ; let $n_T(\rho, c)$ be the number of all the c-points, with due count of multiplicity, of $T(\lambda)$ in the domain $\Delta_\rho\{\lambda: \rho<|\lambda|<\infty\}$; let

$$N_T(\rho, c)=\int_{\rho}^{\infty}\frac{n_T(r, c)}{r}dr \qquad (\sigma<\rho<\infty);$$

and let

$$m_T(\rho, c)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{1}{[T(\rho e^{-it}), c]}dt \qquad (\sigma<\rho\leqq\infty),$$

where

$$[T(\rho e^{-it}), c]=\frac{|T(\rho e^{-it})-c|}{\sqrt{(1+|T(\rho e^{-it})|^2)(1+|c|^2)}}.$$

Then the equality

$$N_T(\rho, c)+m_T(\rho, c)-m_T(\infty, c)+\log\sqrt{1+|\xi|^2}$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\log\sqrt{1+|T(\rho e^{-it})|^2}\,dt$$

holds for every ρ with $\sigma<\rho<\infty$ and every finite value c different from ξ; and here the left and right sides both converge to $\log\sqrt{1+|\xi|^2}$ as ρ becomes infinite.

Proof. If we now consider the function $f(\lambda)\equiv T\left(\frac{\rho^2}{\lambda}\right)=\xi+\chi\left(\frac{\rho^2}{\lambda}\right)$, then by reasoning exactly like that applied to prove Theorem 46 we obtain

$$\log|T(\infty)-c|+N_T(\rho, c)=\frac{1}{2\pi}\int_0^{2\pi}\log|T(\rho e^{-it})-c|\,dt \qquad (\sigma<\rho<\infty),$$

where $T(\infty)=\xi$ by the definition of $\chi(\lambda)$; and the desired equality in the statement of the present theorem is readily deduced from the just established result. In addition, it is at once obvious that the left and right sides of the desired equality both converge to $\log\sqrt{1+|\xi|^2}$ as ρ becomes infinite.

If, in the hypotheses of Theorem 49, c assumes the value ξ, then the following theorem is valid.

Theorem 50. Let $T(\lambda)$, σ, and ξ have the same meanings as those in Theorem 49 respectively; and let

$$\tilde{m}_T(\rho, \xi)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{|C_{-k}|}{[T(\rho e^{-it}), \xi]\rho^k}\,dt \qquad (\sigma<\rho<\infty),$$

on the assumption that the coefficients $C_{-\mu}$ in the expansion of $\chi(\lambda)$ satisfy the hypotheses given in the statement of Theorem 47. Then the equality

$$N_T(\rho, \xi)+\tilde{m}_T(\rho, \xi)-\log\sqrt{1+|\xi|^2}=\frac{1}{2\pi}\int_0^{2\pi}\log\sqrt{1+|T(\rho e^{-it})|^2}\,dt$$

holds for every ρ with $\sigma<\rho<\infty$ and its left and right sides both converge to $\log\sqrt{1+|\xi|^2}$ as ρ becomes infinite.

Proof. According to Theorem 47,

$$\tilde{N}(\rho, 0)+\tilde{m}(\rho, 0)=\frac{1}{2\pi}\int_0^{2\pi}\log\sqrt{1+|\chi(\rho e^{-it})|^2}\,dt$$

and here it is easily found that $\tilde{N}(\rho, 0)=N_T(\rho, \xi)$ for every ρ with $\sigma<\rho<\infty$. Consequently we have

$$N_T(\rho, \xi)+\frac{1}{2\pi}\int_0^{2\pi}\log\frac{1}{[T(\rho e^{-it})-\xi, 0]}\,dt+\log|C_{-k}|-k\log\rho$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\log\sqrt{1+|T(\rho e^{-it})-\xi|^2}\,dt,$$

so that

$$N_T(\rho, \xi)-\frac{1}{2\pi}\int_0^{2\pi}\log|T(\rho e^{-it})-\xi|\,dt+\log|C_{-k}|-k\log\rho=0.$$

By adding $\frac{1}{2\pi}\int_0^{2\pi}\log\sqrt{1+|T(\rho e^{-it})|^2}\,dt+\log\sqrt{1+|\xi|^2}$ to the left and right sides of the last equality, we can easily establish the desired equality in the statement of the present theorem. Since, moreover, $T(\rho e^{-it})-\xi=\sum_{\mu=k}^{\infty}C_{-\mu}(\rho e^{-it})^{-\mu}$, it is verified by direct computation that

$$\tilde{m}_T(\rho, \xi)\to 2\log\sqrt{1+|\xi|^2} \qquad (\rho\to\infty),$$

and hence the left and right sides of the desired equality both tend to $\log\sqrt{1+|\xi|^2}$ as ρ becomes infinite.

Theorem 51. Let $T(\lambda)$ and σ be the same notations as above; let the ordinary part of $T(\lambda)$ be a finite constant ξ, not 0; and let

$$\hat{T}_{\xi}(\rho)=\frac{1}{2\pi}\int_{0}^{2\pi}\log\sqrt{1+|T(\rho e^{-it})|^{2}}\,dt.$$

Then $\hat{T}_{\xi}(\rho)$ is not only a monotone decreasing function of ρ but also a convex function of $\log\rho$ in the interval $\sigma<\rho<\infty$.

Proof. Let c denote any finite value; let $\hat{n}_{T}(\rho, c)$ denote the number of c-points, with due count of multiplicity, of $T(\lambda)$ in the domain $D_{\rho}\{\lambda: \rho<|\lambda|\leqq\infty\}$ with $\sigma<\rho<\infty$; let $\hat{n}_{T}(\infty, c)$ denote the number of c-points of $T(\lambda)$ at $\lambda=\infty$, that is, let $\hat{n}_{T}(\infty, c)$ be k or 0 according as c equals ξ or not, on the assumption that the coefficients $C_{-\mu}$ in the expansion of $\chi(\lambda)$ satisfy the hypotheses given in the statement of Theorem 47, that is, on the assumption that the point at infinity is a zero-point of order k of $\chi(\lambda)$; let $C_{-\hat{n}_{T}(\infty,c)}$ denote C_{-k} or 1 according as c equals ξ or not; and let

$$\hat{N}_{T}(\rho, c)=\int_{\rho}^{\infty}\frac{\hat{n}_{T}(r, c)-\hat{n}_{T}(\infty, c)}{r}\,dr-\hat{n}_{T}(\infty, c)\log\rho \qquad (\sigma<\rho<\infty),$$

$$\hat{P}(c)=\log|C_{-\hat{n}_{T}(\infty,c)}|+\log\frac{|\xi-c|^{[1-k^{-1}\hat{n}_{T}(\infty,c)]}}{\sqrt{1+|c|^{2}}},$$

where we may and do assume that $|\xi-c|^{[1-k^{-1}\hat{n}_{T}(\infty,c)]}$ equals 1 at $c=\xi$. Then there is no difficulty in showing that

$$\begin{aligned}&\hat{N}_{T}(\rho, c)+m_{T}(\rho, c)+\hat{P}(c)\\&=\begin{cases}N_{T}(\rho, c)+m_{T}(\rho, c)-m_{T}(\infty, c)+\log\sqrt{1+|\xi|^{2}} & (c\neq\xi)\\ N_{T}(\rho, \xi)+\tilde{m}_{T}(\rho, \xi)-\log\sqrt{1+|\xi|^{2}} & (c=\xi).\end{cases}\end{aligned}$$

By means of Theorems 49 and 50, we can therefore verify without difficulty that

$$\hat{T}_{\xi}(\rho)=\frac{1}{\pi}\iint_{A}\hat{N}_{T}(\rho, c)d\omega(c)+\frac{1}{\pi}\iint_{A}m_{T}(\rho, c)d\omega(c)+\frac{1}{\pi}\iint_{A}\hat{P}(c)d\omega(c)$$

where $\frac{1}{\pi}\iint_{A}m_{T}(\rho,c)d\omega(c)$ is a positive constant irrespective of ρ and T which are contained in $m_{T}(\rho, c)$. Putting $S_{T}(\rho)=\frac{1}{\pi}\iint_{A}\hat{n}_{T}(\rho, c)d\omega(c)$, it is found immediately from this equality that $\hat{T}'_{\xi}(\rho)=-\frac{S_{T}(\rho)}{\rho}<0$ for $\sigma<\rho<\infty$. Since, in addition, $\hat{T}_{\xi}(\rho)\to\log\sqrt{1+|\xi|^{2}}$ $(\rho\to\infty)$, we obtain

$$\hat{T}_{\xi}(\rho)=\int_{\rho}^{\infty}\frac{S_{T}(\rho)}{\rho}\,d\rho+\log\sqrt{1+|\xi|^{2}}$$

and $\frac{d^{2}\hat{T}_{\xi}(\rho)}{d(\log\rho)^{2}}=-S'_{T}(\rho)\rho$. Here $S_{T}(\rho)$ is a monotone decreasing function of ρ in the interval $\sigma<\rho<\infty$ in accordance with the definition of $\hat{n}_{T}(\rho, c)$, so that $\frac{d^{2}\hat{T}_{\xi}(\rho)}{d(\log\rho)^{2}}\geqq 0$ in that interval.

The theorem has thus been proved.