197. *On Axiom Systems of Propositional Calculi. XII*

By Yoshinari ARAI

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Our purpose in this paper is twofold: first, to prove that Lukasiewicz second axiom system of propositional calculus implies his first axioms, and second, to show that the axiom of (L_2)-system derives (F), (H), (L_3), (M), (R), (S_1), and (S_2) axiom systems. For the notations and rules of inference, see [1]. The fundamental axioms are the following three theses:

1 $CCCpqrCNpr$,
2 $CCCpqrCqr$,
3 $CCNprCCqrCCpqr$.

We shall first give a proof of $(L_2)\Rightarrow(L_1)$. From the (L_2)-system, we have the following theses:

2 p/Crq, q/p, r/Cqp $*C2$ p/r, r/p—4,

4 $CpCqp$.

1 q/p, $r/CsCpp$ $*C4$ p/Cpp, q/s—5,

5 $CNpCsCpp$.

3 q/Cpp, $r/CCpCqpCpp$ $*C5$ $s/CpCqp$—$C4$ p/Cpp, $q/CpCqp$—$C4$ q/p—$C4$—6,

6 Cpp.

1 r/Cpq $*C6$ p/Cpq—7,

7 $CNpCpq$.

3 r/Cpr $*C7$ q/r—8,

8 $CCqCprCCpqCpr$.

1 p/q, q/Cpr, $r/CCpqCpr$ $*C8$—9,

9 $CNqCCpqCpr$.

2 q/r, $r/CCpqCpr$ $*C4$ p/Cpr, q/Cpq—10,

10 $CrCCpqCpr$.

3 p/q, q/r, $r/CCpqCpr$ $*C9$—$C10$—11,

11 $CCqrCCpqCpr$.

2 p/r, q/p, $r/CqCrp$ $*C4$ p/Crp—12,

12 $CpCqCrp$.

Theses 7, 12 and the axiom 1 of (L_2) are the axioms of (S_1)-system.

Our first proof, say, proving to deduce (L_1)-system from (L_2) axioms, would be run:

8 p/q, q/p, r/Crp $*C12$—$C13$,

13 $CCqpCqCrp$.

13 p/Cpq, q/Np, r/s *$C7$—14,

14 $CNpCsCpq$.

3 q/Cqr, $r/CqCpr$ *$C14$ q/r, s/q—$C13$ p/r, r/p—15,

15 $CCpCqrCqCpr$.

15 p/Cqr, q/Cpq, r/Cpr *$C11$—16,

16 $CCpqCCqrCpr$.

15 p/Np, q/p, r/q *$C7$—17,

17 $CpCNpq$.

3 q/p, $r/CNNpp$ *$C17$ p/Np, q/p—$C4$ q/NNp—$C6$—18,

18 $CNNpp$.

3 p/Np, q/p, r/p *$C18$—$C6$—19,

19 $CCNppp$.

Theses 16, 17, and 19 are the axioms of (L_1)-system. Then it follows that (L_1)-system is equivalent to (L_2)-system.

Next we shall prove that (L_2) axioms imply (F), (H), (L_3), (M), (R), and (S_2) axiom systems.

3 q/Nq, $r/CqNp$ *$C4$ p/Np—$C7$ p/q, q/Np—20,

20 $CCpNqCqNp$.

11 p/s, q/NNp, r/p *$C18$—21,

21 $CCsNNpCsp$.

11 $p/CNpNq$, $q/CqNNp$, r/Cqp *$C21$ s/q—$C20$ p/Np—22,

22 $CCNpNqCqp$.

3 q/Np, r/Np *$C6$ p/Np—$C6$ p/Np—23,

23 $CCpNpNp$.

3 q/Cqr, $r/CCpqCpr$ *$C14$ q/r, s/Cpq—$C11$—24,

24 $CCpCqrCCpqCpr$.

11 p/s, $q/CNpr$, $r/CCqrCCpqr$ *$C3$—25,

25 $CCsCNprCsCCqrCCpqr$.

16 p/NNp, q/p, r/q *$C18$—26,

26 $CCpqCNNpq$.

25 p/Np, s/Cpr *$C26$ q/r—27,

27 $CCprCCqrCCNpqr$.

27 p/q, q/p, r/q *$C6$ p/q—28,

28 $CCpqCCNqpq$.

11 $p/CNqNp$, q/Cpq, $r/CCNqpq$ *$C28$—$C22$ p/q, q/p—29,

29 $CCNqNpCCNqpq$.

15 $p/CNpq$, q/Cqq, $r/CCpqq$ *$C3$ r/q—$C6$ p/q—30,

30 $CCNpqCCpqq$.

15 $p/CNpq$, q/Cpq, r/q *$C30$—31,

31 $CCpqCCNpqq$.

16 p/Cpq, $q/CCqrCpr$, r/s $*C16$—32,

32 $CCCCqrCprsCCpqs$.

32 q/Cqr, r/Csr, $s/CCsqCpCsr$ $*C32$ p/s, $s/CpCsr$—33,

33 $CCpCqrCCsqCpCsr$.

33 p/Nq, s/p $*C7$ p/q, q/r—34,

34 $CCpqCNqCpr$.

20 p/Np, q/p $*C6$ p/Np—35,

35 $CpNNp$.

11 r/NNq $*C35$ p/q—36,

36 $CCpqCpNNq$.

16 p/Cpq, $q/CpNNq$, $r/CNqNp$ $*C36$—$C20$ q/Nq—37,

37 $CCpqCNqNp$.

Hence we can find that theses 4, 18, 24, 35, and 37 appear as five axioms of (F)-system, that theses 4, 11, 15, 17, and 31 appear as five axioms of (H)-system, that theses 4, 22, and 24 appear as three axioms of (L_3)-system, that theses 4, 24, and 29 appear as three axioms of (M)-system, that theses 4, 15, 16, 18, 20, and 23 appear as six axioms of (R)-system, and that theses 12, 30, and 34 appear as three axioms of (S_2)-system.

Reference

[1] Y. Imai and K. Iséki: On axiom systems of propositional calculi. I. Proc. Japan Acad., **41**, 436-439 (1965).