11. A Note on Riemann's Period Relations. II

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1. **Basic notations.** Let W be a Riemann surface of infinite genus and (F_n) its exhaustion in Noshiro's sense (Noshiro [11]), then there exists on W a canonical homology basis of A-type with respect to (F_n) such that $A_1, B_1, \cdots, A_{k(n)}, B_{k(n)}$ form a canonical homology basis of F_n (mod ∂F_n) and $A_i \times B_j = \delta_i^j$, $A_i \times A_j = B_i \times B_j = 0$ (Ahlfors [3]). We denote such a basis by C.H.B. $(F_n)_A$. Especially when (F_n) is a canonical exhaustion, there exists a *special* C.H.B. $(F_n)_A$ which satisfies the following condition: *the cycles A_k, B_k with $k > k(n)$ lie outside of F_n for all n (Ahlfors-Sario [4])*. We denote such a special basis by C.H.B. $(F_n)_B^q$ and call it *a canonical homology basis of B-type* with respect to the exhaustion $(F_n)^q$, where $(F_n)^q$ denotes a canonical exhaustion of W.

Definition 1. *Let Γ_1, Γ_2 be two subspaces in Γ_h. We will say that the generalized bilinear relations between Γ_1 and Γ_2 hold with respect to (F_n) and C.H.B. $(F_n)_A$ if we have for all $\omega_1 \in \Gamma_1$ and all $\omega_2 \in \Gamma_2$*

$$(\omega_1, \omega_2^*) = \lim_{n\to\infty} \sum_{k=1}^{k(n)} \left(\int_{A_k} \omega_1 \int_{B_k} \bar{\omega}_2 - \int_{A_k} \bar{\omega}_2 \int_{B_k} \omega_1 \right). \tag{1.1}$$

Analogously we will say that the special bilinear relations between Γ_1 and Γ_2 hold if we have (1.1) *for $\omega_1 \in \Gamma_1$ with only a finite number of non zero periods.*

2. **Special bilinear relation.** Let $\sigma(A_k), \sigma(B_k)$ be reproducing differentials of class Γ_{h0} associated with cycles A_k, B_k respectively, and let $\tilde{\sigma}(A_k), \tilde{\sigma}(B_k)$ be regular distinguished reproducing differentials of class $\Gamma_{h0} \cap \Gamma_{hse}^*$ associated with cycles A_k, B_k respectively (Ahlfors-Sario [4]). For $\omega_1 \in \Gamma_1$ with only a finite number of non zero periods we define $T\omega_1$ and $\tilde{T}\omega_1$ as follows:

$$T\omega_1 = \sum_{k=1}^{\infty} b_k \sigma(A_k) - a_k \sigma(B_k), \qquad \text{(a finite sum)} \tag{2.1}$$

$$\tilde{T}\omega_1 = \sum_{k=1}^{\infty} b_k \tilde{\sigma}(A_k) - a_k \tilde{\sigma}(B_k), \qquad \text{(a finite sum)} \tag{2.2}$$

where $(A_i, B_i) =$ C.H.B. $(F_n)_A$, $a_k = \int_{A_k} \omega_1$, $b_k = \int_{B_k} \omega_1$.

Theorem 1. *The maximal class of Γ_2 such that the special bilinear relations between $\Gamma_1 = \Gamma_{h0}$ and Γ_2 hold, is closure $(\Gamma_{h0} + \Gamma_{he})$.*

Proof. We put $max\, \Gamma_2 = \Gamma'$. From the assumption we have for arbitrary $\omega_1 \in \Gamma_1 = \Gamma_{h0}$ and arbitrary $\omega_2 \in \Gamma'$

$$(\omega_1, \omega_2^*)=\sum_k \left(\int_{A_k}\omega_1\int_{B_k}\bar{\omega}_2-\int_{A_k}\bar{\omega}_2\int_{B_k}\omega_1\right)+(\omega_1-T\omega_1, \omega_2^*)$$
$$=\sum_k \left(\int_{A_k}\omega_1\int_{B_k}\bar{\omega}_2-\int_{A_k}\bar{\omega}_2\int_{B_k}\omega_1\right). \tag{2.3}$$

On the other hand from (2.1) we obtain $\omega_1-T\omega_1\in\Gamma_{he}\cap\Gamma_{h0}$. Therefore, for $\omega_2\in\Gamma'$, we have $\omega_2^*\in(\Gamma_{he}\cap\Gamma_{h0})^{\perp}$, hence $\Gamma'\subset$ closure $(\Gamma_{h0}+\Gamma_{he})$. Conversely, for $\omega_2\in$ closure $(\Gamma_{h0}+\Gamma_{he})$, we have $(\omega_1-T\omega_1, \omega_2^*)=0$, because $\omega_1-T\omega_1\in\Gamma_{h0}\cap\Gamma_{he}$. Therefore we get $\Gamma'\supset$ closure $(\Gamma_{h0}+\Gamma_{he})$.

Corollary 1. (Accola [1]). *The validity of the special bilinear relations between* $\Gamma_1=\Gamma_{h0}$ *and* $\Gamma_2=\Gamma_{hse}$ *is equivalent to* $\Gamma_{hm}=\Gamma_{h0}\cap\Gamma_{he}$.

Proof. From Theorem 1 we can put $\Gamma_{hse}=$ closure $(\Gamma_{h0}+\Gamma_{he})$ and therefore we have $\Gamma_{hm}=\Gamma_{h0}\cap\Gamma_{he}$. Conversely if $\Gamma_{hm}=\Gamma_{h0}\cap\Gamma_{he}$, we have closure $(\Gamma_{h0}+\Gamma_{he})=\Gamma_{hse}$ and from Theorem 1 special bilinear relations between $\Gamma_1=\Gamma_{h0}$ and $\Gamma_2=\Gamma_{hse}$ hold.

Theorem 2. *The maximal class of* Γ_2 *such that the special bilinear relations between* $\Gamma_1=\Gamma_{hse}$ *and* Γ_2 *hold, is* Γ_{h0}.

Proof. We put $\Gamma'=max\,\Gamma_2$. From (2.1), (2.3) and the assumption we have for $\omega_2\in\Gamma'$ $\omega_2^*\perp\Gamma_{he}$, because $\omega_1-T\omega_1\in\Gamma_{he}$, therefore $\Gamma'\subset\Gamma_{h0}$. Conversely we can get $\Gamma_{h0}\subset\Gamma'$ (Accola [1] or Ahlfors-Sario [4]). Consequently we have $\Gamma_{h0}=\Gamma'$.

Corollary 2. (Mori [10]). *A Riemann surface* W *is of class* O_{KD} *if and only if the special bilinear relations between* $\Gamma_1=\Gamma_{hse}$ *and* $\Gamma_2=\Gamma_{hse}$ *hold.*

We can prove this by the same way as in Corollary 1.

Remark 1. A Riemann surface which satisfies the condition of Theorem 1 in Matsui [9] or Theorem 1 in Kobori and Sainouchi [6] belongs to O_{KD}.

Theorem 3. *The maximal class of* Γ_2 *such that the special bilinear relations between* $\Gamma_1=\Gamma_{hs}\cap\Gamma_{hse}$ *and* Γ_2 *hold, is* $\Gamma_{h0}\dotplus\Gamma_{he}\cap\Gamma_{he}{}^*$.

Proof. From (2.1), (2.3) and the assumption we have for $\omega_2\in\Gamma'=max\,\Gamma_2$ $\omega_2\perp\Gamma_{hs}{}^*\cap\Gamma_{he}{}^*$, because $\omega_1-T\omega_1\in\Gamma_{hs}\cap\Gamma_{he}$. Therefore we get $\Gamma'\subset\Gamma_{h0}\dotplus\Gamma_{he}\cap\Gamma_{he}{}^*$. Conversely for $\omega_2\in\Gamma_{h0}\dotplus\Gamma_{he}\cap\Gamma_{he}{}^*$ we have

$$(\omega_1-T\omega_1, \omega_2^*)=0, \text{ because } \Gamma_{hs}{}^*\cap\Gamma_{he}{}^*\perp(\Gamma_{h0}\dotplus\Gamma_{he}\cap\Gamma_{he}{}^*).$$

Therefore we have $\Gamma'\supset\Gamma_{h0}\dotplus\Gamma_{he}\cap\Gamma_{he}{}^*$.

Corollary 3. (Mori [10]). *The validity of the special bilinear relations between* $\Gamma_1=\Gamma_{hs}\cap\Gamma_{hse}$ *and* $\Gamma_2=\Gamma_{hse}$ *is equivalent to* $\Gamma_{he}\cap\Gamma_{hse}{}^*\subset\Gamma_{he}{}^*$.

Theorem 4. *The maximal class of* Γ_2 *such that the special bilinear relations between* $\Gamma_1=\Gamma_{hse}\cap\Gamma_{h0}{}^*$ *and* Γ_2 *hold, includes the class*

$$(closure\,(\Gamma_{h0}+\Gamma_{hN}\cap\Gamma_{hN}^*))\cap\Gamma_{hse}$$

where Γ_{hN} *is the orthogonal complement of* $\Gamma_{hse}\cap\Gamma_{h0}^*$ *in* Γ_h.

Proof. From (2.2), (2.3) and the assumption we get

$$\omega_1-\tilde{T}\omega_1\in\Gamma_{he}\cap(\Gamma_1+\Gamma_1^*).$$

On the other hand

$$\Gamma_{hse}\cap(\text{closure}\,(\Gamma_{h0}+\Gamma_{hN}\cap\Gamma_{hN}^*))$$
$$=\Gamma_{hse}\cap(\Gamma_{h0}^{\perp}\cap(\Gamma_1^{\perp}\cap\Gamma_1^{\perp *})^{\perp})^{\perp}\subset\Gamma_{hse}\cap(\Gamma_{he}^*\cap(\Gamma_1+\Gamma_1^*))^{\perp}.$$

Therefore, for $\omega_2\in(\Gamma_{hse}\cap\text{closure}\,(\Gamma_{h0}+\Gamma_{hN}\cap\Gamma_{hN}^*))$, we have

$$(\omega_1-\tilde{T}\omega_1,\ \omega_2^*)=0.$$

Consequently $max\,\Gamma_2$ includes the class $\Gamma_{hse}\cap(\text{closure}(\Gamma_{h0}+\Gamma_{hN}\cap\Gamma_{hN}^*))$.

3. **Generalized bilinear relation.** Let C be a cycle on Ω which is an elementary domain on W (Kusunoki [8]), $(C)_\Omega$ a family of rectifiable curves on Ω which are homologous to C on Ω, and let (C) be a family of rectifiable curves which are homologous to C on W.

Lemma 1. (Hersch [5], Accola [2], and Kusunoki [7])

$$\left.\begin{array}{l}\lambda(A_k)\lambda(B_k)\geqslant 1,\ \lambda(A_k)_\Omega\lambda(B_k)_\Omega\geqslant 1.\\ \|\sigma_\Omega(A_k)\|_\Omega^2\leqslant\lambda(A_k)_\Omega,\ \|\sigma_\Omega(B_k)\|_\Omega^2\leqslant\lambda(B_k)_\Omega\end{array}\right\}\qquad(3.1)$$

where $\lambda(C)_\Omega$ is the extremal length of $(C)_\Omega$ and $\sigma_\Omega(A_k)$, $\sigma_\Omega(B_k)$ are reproducing differentials on Ω associated with the cycles A_k, B_k respectively.

Let us consider an open Riemann surface W of infinite genus and its canonical exhaustion $(F_n)^q$, and for each n we define $T_{F_n}\omega_2$ as follows:

$$T_{F_n}\omega_2=\sum_{k=1}^{k(n)}b_k\sigma_{F_n}(A_k)-a_k\sigma_{F_n}(B_k)\qquad(3.2),$$

where $(A_i, B_i)=$C.H.B. $(F_n)_A^q$, $a_i=\int_{A_i}\omega_2$ and $b_i=\int_{B_i}\omega_2$.

Lemma 2. *Let* $(A_i, B_i)=C.H.B.\,(F_n)_A^q$ $((F_n)^q$: *a canonical exhaustion). Choose* $\omega_1\in\Gamma_{h0}$ *and* $\omega_2\in\Gamma_{hse}$, *and decompose* ω_1 *such that* $\omega_1=\omega_{1h0F_n}+\omega_{1heF_n^*}$ *on* F_n. *Then the relation*

$$(\omega_1,\omega_2^*)=\lim_{n\to\infty}\sum_{k=1}^{k(n)}\left(\int_{A_k}\omega_1\int_{B_k}\bar{\omega}_2-\int_{A_k}\bar{\omega}_2\int_{B_k}\omega_1\right)\qquad(3.3)$$

is true if and only if

$$|(\omega_{1heF_n^*},(T_{F_n}\omega_2)^*)_{F_n}|\to 0 \text{ as } F_n\to W.\qquad(3.4)$$

Proof. We have $(\omega_1,\omega_2^*)_{W-F_n}\to 0$ as $F_n\to W$ and from (3.2) $\omega_2-T_{F_n}\omega_2\in\Gamma_{he}(F_n)$ where $\Gamma_{he}(F_n)$ is the class of harmonic exact differentials on F_n. Therefore we have

$$(\omega_1,\omega_2^*-(T_{F_n}\omega_2)^*)_{F_n}=(\omega_{1heF_n^*},\omega_2^*-(T_{F_n}\omega_2)^*)_{F_n}.$$

Consequently from the assumption it follows that

$$\begin{aligned}(\omega_1,\omega_2^*)&=\lim\,(\omega_1,\omega_2^*)_{W-F_n}+\lim\,(\omega_1,(T_{F_n}\omega_2)^*)_{F_n}\\&\quad+\lim\,(\omega_{1heF_n^*},\omega_2^*)_{F_n}-\lim\,(\omega_{1heF_n^*},(T_{F_n}\omega_2)^*)_{F_n}\\&=\lim\,(\omega_1,(T_{F_n}\omega_2)^*)_{F_n},\qquad\text{q.e.d.}\end{aligned}$$

Theorem 5. *If there exists an exhaustion* $(F_n)^q$ *and a C.H.B.* $(F_n)^q_B$ *which satisfy the following conditions*:

$$\sum_{A_k, B_k \subset F^i_{n-1}} \sqrt{\lambda(A_k)_{F^i_{n-1}} \lambda(B_k)_{F^i_{n-1}}} \leqslant K \quad (\textit{for all } n \textit{ and } i) \tag{3.5}$$

where $(A_k, B_k) = C.H.B.\,(F_n)^q_B$ *and* $F_n - \bar{F}_{n-1} = \sum F^i_{n-1}$ (F^i_{n-1}: *a component*), *then the generalized bilinear relations between* Γ_{h0} *and* Γ_{hse} *hold with respect to* $(F_n)^q$ *and C.H.B.* $(F_n)^q_B$.

Proof. From the assumption and the Schwarz's inequality we have

$$\left| \sum_{A_k \cdot B_k \subset F^i_{n-1}} \bar{b}_k \int_{A_k} \omega_{1heF^*_n} - \bar{a}_k \int_{B_k} \omega_{1heF^*_n} \right|$$
$$\leqslant \sum_{A_k, B_k \subset F^i_{n-1}} \| \omega_2 \|_{F^i_{n-1}} \| \omega_{1heF_n} \|_{F^i_{n-1}} \sqrt{\lambda(A_k)_{F^i_{n-1}} \lambda(B_k)_{F^i_{n-1}}}$$
$$\leqslant K \| \omega_2 \|_{F^i_{n-1}} \| \omega_{1heF_n} \|_{F^i_{n-1}}.$$

Therefore we get

$$| (\omega_{1heF^*_n}, (T_{F_n}\omega_2)^*) F_n | \leqslant K \sum_k^n \| \omega_2 \|_{F_k - F_{k-1}} \| \omega_{1heF_n} \|_{F_k - F_{k-1}}$$
$$\leqslant K \Big(\sum_k^n \| \omega_2 \|^2_{F_k - F_{k-1}} \Big)^{\frac{1}{2}} \cdot \Big(\sum_k^n \| \omega_{1heF_n} \|^2_{F_k - F_{k-1}} \Big)^{\frac{1}{2}}$$
$$\leqslant K \| \omega_2 \|_{F_n} \| \omega_{1heF_n} \|_{F_n}.$$

On the other hand $\| \omega_{1heF_n} \|_{F_n} \to 0$ as $F_n \to W$ (Ahlfors-Sario [4]), we obtain

$$| (\omega_{1heF^*_n}, (T_{F_n}\omega_2)^*)_{F_n} | \to 0 \text{ as } F_n \to W.$$

Therefore from Lemma 2 the proof is complete.

Corollary 4. *If there exists a* $(F_n)^q$ *and a C.H.B.* $(F_n)^q_B$ *such that for all n and i*

$$\left. \begin{array}{l} \lambda(A_k)_{F^i_{n-1}} \leqslant K, \qquad \lambda(B_k)_{F^i_{n-1}} \leqslant K \\ \textit{the genus of } F^i_{n-1} = p^i_{n-1} \leqslant P, \end{array} \right\} \tag{3.6}$$

then the generalized bilinear relations between Γ_{h0} *and* Γ_{hse} *hold with respect to* $(F_n)^q$ *and C.H.B.* $(F_n)^q_B$.

Remark 2. If the condition (3.6) is satisfied, we can get easily for all $\omega \in \Gamma_{hse}$

$$\sum | a_k |^2 + | b_k |^2 < \infty$$

where $a_k = \int_{A_k} \omega$, $b_k = \int_{B_k} \omega$. Analogously we can get for all $\omega_1 \in \Gamma_{h0}$

$$\lim_{n\to\infty} \Big(\sum_{k=1}^{k(n)} \Big(\Big| \int_{A_k} \omega_{1heF^*_n} \Big|^2 + \Big| \int_{B_k} \omega_{1heF^*_n} \Big|^2 \Big) \Big) = 0.$$

Remark 3. We can construct an example of the surface which satisfies (3.6) and does not belong to O_{AD}. But in the present step it is not sure that there exist surfaces which belong to O'' (Kusunoki [7]) and do not satisfy (3.6).

Corollary 5. *If the condition* (3.5) *is satisfied, from Corollary* 1 *we have* $\Gamma_{hm} = \Gamma_{h0} \cap \Gamma_{he}$.

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