8. *Some Theorems in B-algebra*

By Kiyoshi ISÉKI

(Comm. by Kinjirô KUNUGI, M.J.A., Jan. 12, 1966)

In my notes ([1], [2], and [3]), I gave algebraic formulations of classical propositional calculus, and some characterization of Boolean algebra. In this note, I shall give proofs of some results in classical propositional calculus by view of algebraic formulations.

Let $M=\langle X, 0, *, \sim\rangle$ be a B-algebra, i.e., M satisfies the following axioms:

1 $x*y\leqslant x$,

2 $(x*z)*(y*z)\leqslant(x*y)*z$,

3 $x*y\leqslant(\sim y)*(\sim x)$,

4 $0\leqslant x$,

5 $x\leqslant y$ and $y\leqslant x$ imply $x=y$,

where $x\leqslant y$ means $x*y=0$.

Theorem 1. *In a B-algebra* M, *we have*

$$(x*u)*(x*z)\leqslant z*(y*x).$$

The formula in theorem 1 is used to give axioms of classical propositional calculus by A. Rose [6]. For proofs, we freely use some results mentioned in my notes [1], [2], and [3]. We refer, for example, the proposition (16) in my note [2] as ((16) in [2]).

Proof.
$$\begin{aligned}(x*u)*(x*z)&=(x*(x*z))*u && \text{((1) in [2])}\\ &\leqslant(z*(\sim x))*u && \text{((13) in [3])}\\ &=(z*u)*(\sim x) && \text{((1) in [2])}\\ &\leqslant z*(\sim x). && \text{((5) in [2])}\end{aligned}$$

On the other hand, we have $y*x\leqslant\sim x$ from ((8) in [1]). Hence by ((5) in [2]), we have $z*(\sim x)\leqslant z*(y*x)$. Therefore we have

$$(x*u)*(x*z)\leqslant z*(y*x).$$

The following is an algebraic formulation of a formula given by C. Meredith (see A. N. Prior [5]).

Theorem 2. *In a B-algebra* M, *we have*

$$(x*s)*(x*t)\leqslant t*(z*(((\sim s)*(\sim z))*(y*x))).$$

Proof. $(x*s)*(x*t)=((\sim s)*(\sim x)*((\sim t)*(\sim x)\leqslant((\sim s)*(\sim t))$ $*(\sim x)=((\sim s)*(\sim x))*(\sim t)=(x*s)*(\sim t)=t*\sim(x*s)$.

Further, by $y*(x*(\sim y))\leqslant y*x$ in [3] and $x*y\leqslant\sim y$, we have

$$z*((x*s)*(\sim z)\leqslant z*(x*s)\leqslant\sim(x*s).$$

Hence

$$\begin{aligned}t*(\sim(x*s))&\leqslant t*(z*((x*s)*(\sim z)\\ &=t*(z*((\sim s*\sim x)*(\sim y))\\ &=t*(z*((\sim s*\sim z)*(\sim x)).\end{aligned}$$

Applying the well known formula $y * x \leqslant \sim x$ and ((5) in [2]), we have

$$t * (z * ((\sim s * \sim z) * (\sim x)) \\ \leqslant t * (z * ((\sim s * \sim z) * (y * x))$$

and we have

$$(x * s) * (x * t) \\ \leqslant t * (z * (((\sim s) * (\sim z)) * (y * x))).$$

Next we shall consider a formula given in J. Lukasiewicz and A. Tarski [4], then we have

Theorem 3. *In a B-algebra, we have*

$$x * w \leqslant (x * ((((u * t) * (s * t)) * ((u * s) * r)) \\ * (((\sim t) * s) * (\sim r)))) * ((y * z) * y)).$$

Proof. Clearly $y * z \leqslant y$, i.e. $(y * z) * y = 0$. We shall prove

$$(((u * t) * (s * t)) * ((u * s) * r)) * (((\sim t) * s) * (\sim r)) = 0,$$

then the right side of the formula is $x*(0*0)=x$ by ((17) in [3]). Hence we have $x * w \leqslant x$, which completes the proof. Now we prove the auxiliary formula:

$$((u * t) * (s * t)) * ((u * s) * r) \\ \leqslant ((u * s) * t) * ((u * s) * r) \\ = (\sim t * \sim (u * s)) * (\sim r * \sim (u * s)) \\ \leqslant (\sim t * \sim r) * \sim (u * s) = (r * t) * \sim (u * s) \\ = (r * \sim (u * s)) * t = (\sim r * (u * s)) * t.$$

Consider $((r * t) * \sim (u * s)) * ((\sim t * s) * r)$, then the expression is equal to

$$((r * t) * \sim (u * s)) * ((\sim t * \sim r) * s) \\ = ((r * t) * \sim (u * s)) * ((r * t) * s) \\ \leqslant (s * \sim (u * s)) * (r * t) \leqslant s * (\sim (u * s)).$$

From ((8) in [1]), we have $u * s \leqslant \sim s$. Hence $s \leqslant \sim (u * s)$. Therefore, from ((5) in [3]), we have $s * (\sim (u * s)) \leqslant s * s = 0$. Hence we have

$$(((u * t) * (s * t)) * ((u * s) * r)) * ((\sim t * s) * (\sim r)) = 0,$$

and we complete the proof of Theorem 3.

Remarks. Theorem 1 is proved in the I-algebra which is a generalisation of B-algebra and corresponds to the implicational propositional calculus. The detail is contained in the former paper by Y. Imai and the present writer. Theorem 2 characterizes the B-algebra. This will be shown in a later paper. To give a proof of Theorem 3 by J. Lukasiewicz and A. Tarski, S. Tanaka used the idea of my above proof. For the detail, see S. Tanaka [7].

References

[1] K. Iséki: Algebraic formulations of propositional calculus. Proc. Japan Acad., **41**, 803–807 (1965).

[2] ——: Axiom systems of B-algebra. Proc. Japan Acad., **41**, 808–811 (1965).

[3] ——: A characterization of Boolean algebra. Proc. Japan Acad., **41**, 893-897 (1965).
[4] J. Lukasiewicz and A. Tarski: Untersuchungen über den Aussagenkalkül. C. R. de Varsovie, Cl. III, **23**, 30-50 (1930).
[5] A. N. Prior: Formal Logic. Oxford (1962).
[6] A. Rose: A formulation of the C-0 propositional calculus. Proc. Cambridge Philos. Soc., **47**, 635-636 (1951).
[7] S. Tanaka: On axiom systems of propositional calculus. XIII. Proc. Japan Acad., **41**, 904-907 (1965).