1. *On Relative Maximal Ideals in Lattices*

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1. Introduction. Let S be a sublattice of a lattice L. An ideal M of L shall be called a *relative maximal ideal* with respect to S, like that in a ring, when M is maximal among ideals which are disjoint to S. It was pointed out by Grätzer and Schmidt [1] that there is a close connection between relative maximal ideals and prime ideals. In the present paper we intend to make some additional researches to them and give an assertion analogous to Cohen's theorem in ideal theory for rings.

Again the properties of relative maximal ideals are useful for the decomposition theories in distributive lattices. So we shall give in § 3 new proofs of Kurosch-Ore Theorem concerning the decomposition of elements, which is generalized by Dilworth and Crawley [4], and Hashimoto's theorem [3] concerning the decomposition of ideals.

2. Relative maximal ideals. Let P be a prime ideal of a lattice L, then the complement $L-P$ of P is a dual prime ideal. So every prime ideal P of a lattice L becomes a relative maximal ideal with respect to a sublattice $L-P$. Concerning the converse we shall show the theorem of Grätzer and Schmidt [1] in a somewhat generalized form.

Theorem 1. *Each of the following conditions are necessary and sufficient in order that a lattice L be distributive;*

(1) *every relative maximal ideal of L is prime;*

(2) *every relative maximal ideal of L with respect to a one-element sublattice is prime.*

Proof. Let M be a relative maximal ideal with respect to a sublattice S of a distributive lattice L. Suppose that M is not prime. Then there exist elements x, y such that $x \notin M$, $y \notin M$, and $x \cap y \in M$. $M \cup (x] \supsetneqq M$ and $M \cup (y] \supsetneqq M$ imply $\{M \cup (x]\} \cap S \ni s_1$ and $\{M \cup (y]\} \cap S \ni s_2$ by the maximality of M, hence $\{M \cup (x]\} \cap \{M \cup (y]\} \ni s_1 \cap s_2$. Since the ideals of a distributive lattice themselves form a distributive lattice, $s_1 \cap s_2 \in \{M \cup (x]\} \cap \{M \cup (y]\} = M \cup \{(x] \cap (y]\} = M \cup (x \cap y] = M$, which is a contradiction. Obviously (1) implies (2), accordingly we need only prove that (2) implies the distributivity of L. If a lattice L is not distributive, there exists in L a sublattice isomorphic to the lattice of Fig. 1 or Fig. 2. But in both cases, the relative maximal ideal with respect to b containing the principal

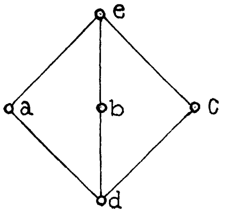

Fig. 1

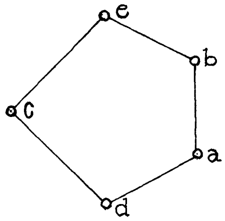

Fig. 2

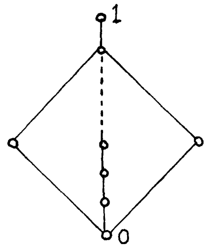

Fig. 3

ideal $(a]$ contains $d=b\cap c$, but does not contain b and c. (The existence of those relative maximal ideals is due to the Axiom of Choice.) It is contrary to the assumption.

Corollary 1. *Every maximal ideal of a distributive lattice is prime.*

It is easy to see that every prime ideal of a relatively complemented lattice is maximal.

Corollary 2. *Every relative maximal ideal of a relatively complemented distributive lattice is maximal.*

Next it has been proved by Cohen [2] that every ideal of a commutative ring R is generated by a finite number of elements if and only if every prime ideal of R is generated by a finite number of elements. Now we shall show the analogous assertion for lattices.

Theorem 2. *If every relative maximal ideal of a lattice L with 1 is principal, then L satisfies the ascending chain condition and hence every ideal of L is principal.*

Proof. Suppose that L contains an infinite ascending chain

$$C:\quad a_1<a_2<a_3<a_4<\cdots,$$

and put $J=\{x;\ x\leqq a_i \text{ for some } a_i\in C\}$, $J'=\{y;\ y\geqq a_i \text{ for all } a_i\in C\}$. J' is non empty since J' contains 1. J and J' are an ideal and a dual ideal respectively, and they are disjoint. Then there exists a relative maximal ideal M which contains J and is disjoint to J'. If we denote $M=(m]$ by the assumption, then $m\geqq a_i$ for all $a_i\in C$ and $m\notin J'$, that is a contradiction.

Corollary. *Every ideal of a distributive lattice L with 1 is principal if and only if every prime ideal of L is principal.*

The distributivity in the corollary can not be dispensed with. The lattice of Fig. 3 does not satisfy the ascending chain condition nevertheless every prime ideal of it is principal.

3. Factorization of ideals. An ideal I of a lattice L is called *factorizable* if and only if it is decomposable into the meet of the prime ideals which contain it. Now let M be a relative maximal ideal of a lattice L with respect to a one-element sublattice $\{a\}$ of

L. If a set $\{I_\alpha\}$ of ideals satisfies $M=\bigcap_\alpha I_\alpha$, then some I_α does not contain a and $M=I_\alpha$, since M is maximal.

Lemma 1. *The relative maximal ideal of a lattice with respect to a one-element sublattice is meet irreducible. Accordingly the relative maximal ideal of a lattice with respect to a one-element sublattice is factorizable if and only if it is prime.*

Then we show another form of Hashimoto's theorem [3].

Theorem 3. *Each of the following conditions are necessary and sufficient in order that a lattice L be distributive;*

(1) *every ideal of L is the meet of the prime ideals which contain it;*

(2) *every relative maximal ideal of L is the meet of the prime ideals which contain it;*

(3) *every relative maximal ideal of L with respect to a one-element sublattice is the meet of the prime ideals which contain it.*

Proof. Suppose that L is a distributive lattice, and let I be an ideal of L such that $I \subsetneqq \bigcap_\alpha P_\alpha$ for all prime ideals P_α $(\alpha \in A)$ such that $P_\alpha \supseteqq I$. Then we can find an element x such that $I \not\ni x$, and $\bigcap_\alpha P_\alpha \ni x$, and there exists a relative maximal ideal with respect to x, containing I, which is prime by Theorem 1. This is a contradiction to $\bigcap_\alpha P_\alpha \ni x$. Hence every ideal of L is the meet of the prime ideals which contain it.

Obviously (1) implies (2) and (2) implies (3). (3) implies the distributivity of L by Theorem 1 and Lemma 1. Thus the proof is completed.

Again an element c of a lattice L is called to be *compact* if $c \leqq \cup S$ implies $c \leqq \cup S'$ for a finite subset S' of S, and a lattice L is said to be *compactly generated* if L is complete and every element of L is a join of compact elements. If every interval $[a, b]$$(a \neq b)$ of a lattice L contains an element covering a, then L is called *atomic*.

Recently, Dilworth and Crawley [4] have shown that the existence and uniqueness theorems for decompositions into irreducibles hold for the compactly generated atomic lattices. Now we shall show a simpler proof of one of them.

Lemma 2. *If b covers a in a distributive lattice L, then there exists one and only one prime ideal which contains a but not b. Further if L is a compactly generated lattice, then that prime ideal is principal.*

Proof. A relative maximal ideal P with respect to b containing a is prime by Theorem 1. Suppose that Q is any prime ideal which contains a and not b, then we have $(q \cup a) \cap b = a \in P$ and $q \leqq$

$q \cup a \in P$ for all $q \in Q$, hence $Q \subseteqq P$, similary $P \subseteqq Q$ and we have $Q=P$. If L is compactly generated, then $b = \cup \{c;\ c \leqq b,$ compact$\}$ and there exists a compact element c such that $b \geqq c$, $a \ngeqq c$. Let P be a relative maximal ideal with respect to c which contains $(a]$ and put $p = \cup P$. If $c \leqq p = \cup P$, then $c \leqq \cup P'$ where P' is a finite subset of P and thus $c \in P$, which is a contradiction. Hence $(p] \supseteqq P \supseteqq (a]$ and $(p] \not\ni c$ imply $P=(p]$ by the maximality of P.

Theorem 4. *Every principal ideal of an atomic, distributive lattice L has a unique irredundant factorization.*

Proof. Let (a] be a principal ideal of a distributive lattice L and B the set of the elements covering a. For any element b of B, there exists one and only one prime ideal P_b which contains $(a]$ and not b, by Lemma 2. Then $(a] = \bigcap_b P_b$; otherwise we can find an element x such that $x \notin (a]$, $x \in \bigcap_b P_b$, and b' such that $x \cup a \geqq b' \succ a$ and $P_{b'} \not\ni x$, which is a contradiction. Now let $(a] = \bigcap_\alpha Q_\alpha$ $(\alpha \in A)$ be another factorization. Since P_b is the unique prime ideal which contains $(a]$ and not b, $P_b \notin \{Q_\alpha\}$ implies $\bigcap_\alpha Q_\alpha \supseteqq (b] \supset (a]$, which is a contradiction, hence $\{Q_\alpha\} \supseteqq \{P_b\}$. Thus the factorization $\bigcap_b P_b$ is irredundant, and if $\bigcap_\alpha Q_\alpha$ is irredundant, then $\{Q_\alpha\} = \{P_b\}$.

And if L is a distributive lattice, then it is known that an element a of L is meet irreducible if and only if $(a]$ is prime. Hence we can deduce the result of Dilworth and Crawley [4].

Corollary. *Every element of a compactly generated, atomic, distributive lattice has a unique irredundant decomposition into irreducibles.*

References

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