33. *Note on the Structure of Regular Semigroups*

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(Comm. by Kinjirô KUNUGI, M.J.A., Feb. 12, 1966)

§1. **Introduction.** Let $X=\{x_1, x_2, \cdots, x_n\}$ be a set in which each element x_i is called a variable. By a *permutation identity* in the variables $x_1, x_2, \cdots, x_n$, we shall mean a form

(P) $$x_1x_2\cdots x_n = x_{p_1}x_{p_2}\cdots x_{p_n}$$

where $(p_1, p_2, \cdots, p_n)$ is a non-trivial permutation of $(1, 2, \cdots, n)$.[1] For example, *commutativity* $x_1x_2=x_2x_1$, *left* [*right*] *normality* $x_1x_2x_3=x_1x_3x_2$ [$x_1x_2x_3=x_2x_1x_3$] and *normality* $x_1x_2x_3x_4=x_1x_3x_2x_4$ are all permutation identities. If a subset M of a semigroup G satisfies the following condition (C.P), then we shall say that M *satisfies* the permutation identity (P) in G:

(C.P) For any mapping φ of X into M, the equality

$$\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)=\varphi(x_{p_1})\varphi(x_{p_2})\cdots\varphi(x_{p_n})$$

is satisfied in G.

In particular if G satisfies the permutation identity (P) in G, we simply say that G satisfies the permutation identity (P). For example, a regular semigroup in which the set of idempotents satisfies commutativity is an inverse semigroup firstly introduced by Vagner [5] under the term 'generalized group' (see also [1], p. 28), and the structure of inverse semigroups was clarified by Preston [3] and [4]. A band (i.e., idempotent semigroup) satisfying [left, right] normality is called a [left, right] normal band, and the structure of [left, right] normal bands was also determined by Kimura and the author [6]. Each of an inverse semigroup and a [left, right] normal band is of course a regular semigroup in which the set of idempotents satisfies a permutation identity. The main purpose of this paper is to present a structure theorem for regular semigroups in each of which the set of idempotents satisfies a permutation identity. The complete proofs are omitted and will be given in detail elsewhere. Any symbol and terminology should be refered to [1] and [6], unless otherwise stated.

§2. **Generalized inverse semigroups.** Let S be a regular semigroup. Then for each element a of S, there exists an element a^* such that $aa^*a=a$ and $a^*aa^*=a^*$. Such an element a^* is called

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1) The form (P) can be considered as the pair $(x_1x_2\cdots x_n, x_{p_1}x_{p_2}\cdots x_{p_n})$ of the two words $x_1x_2\cdots x_n$ and $x_{p_1}x_{p_2}\cdots x_{p_n}$.

an inverse of a. For a given element a of S, an inverse of a is not necessarily unique. An inverse of a is unique for every element a of S if and only if S is an inverse semigroup (see [1], p. 28).[2)]

At first we shall show several lemmas, without proofs.

Lemma 1. (1) *If a regular semigroup S satisfies the following condition* (C), *then the set of idempotents of S is a band*:

(C) *For any elements a, b of S and for any inverses a^* of a and b^* of b, the element b^*a^* is an inverse of ab.*

(2) *If the set of idempotents of a regular semigroup S is a normal band, then S satisfies the condition* (C).

Lemma 2. *If the set B of idempotents of a regular semigroup S satisfies a permutation identity, then B is a normal band.*

Lemma 3. *Let S be a regular semigroup in which the set B of idempotents is a band.*

(1) *If B is a normal band, then the intersection $aS\cap Sb$ $(=aSb)$ of a principal right ideal aS and a principal left ideal Sb is a subsemigroup in which any two of the idempotents commute. In particular, eSe is an inverse semigroup for any idempotent e of S.*

(2) *If every efe, where e, f are elements of B, has a unique inverse having the form ehe, then B is a normal band.*

By using these lemmas, we obtain

Theorem 1. *The following five conditions on a regular semigroup S are equivalent*:

(1) *The set of idempotents of S satisfies a permutation identity in S.*

(2) *The set of idempotents of S is a normal band.*

(3) *The set of idempotents of S is a band, and the intersection $aS\cap Sb$ $(=aSb)$ of a principal right ideal aS and a principal left ideal Sb is a subsemigroup in which any two of the idempotents commute.*

(4) *The set of idempotents of S is a band, and eSe is an inverse subsemigroup for any idempotent e of S.*

(5) *S satisfies the condition* (C). *Further every efe, where e, f are idempotents of S, has a unique inverse having the form ehe.*

By a *generalized inverse semigroup*, hereafter we shall mean a regular semigroup in which the set of idempotents satisfies a permutation identity. Hence it follows from Theorem 1 that a semigroup S is a generalized inverse semigroup if and only if S is regular and the set of idempotents of S is a normal band.

§ 3. A structure theorem. The following is due to McLean [2]:

2) In this case, we denote the inverse of a by a^{-1}.

For any band B, there exist a semilattice (i.e., commutative idempotent semigroup) Γ and a collection of rectangular bands, $\{B_\gamma : \gamma \in \Gamma\}$, such that

(1) $B = \bigcup\{B_\gamma : \gamma \in \Gamma\}$,

(2) $B_\alpha \cap B_\beta = \square$ *for* $\alpha \neq \beta$ *and*

(3) $B_\alpha B_\beta \subset B_{\alpha\beta}$.

Further such a decomposition of B is unique. Accordingly Γ is unique up to isomorphism, and so are the B_γ's.

The Γ above is called the *structure semilattice* of B, and B_γ is called the *γ-kernel* of B. Further, this decomposition is called the *structure decomposition* of B, and denoted by $B \sim \sum\{B_\gamma : \gamma \in \Gamma\}$.

Now, we shall introduce the concept of a *quasi-direct product*: Let Ω be an inverse semigroup, and Γ the set of idempotents of Ω. Then Γ is a commutative idempotent subsemigroup, i.e., a subsemilattice of Ω. Hereafter, we shall call Γ the *basic semilattice* of Ω. Let L and R be a left normal band and a right normal band, having structure decompositions $L \sim \sum\{L_\gamma : \gamma \in \Gamma\}$ and $R \sim \sum\{R_\gamma : \gamma \in \Gamma\}$ respectively. Let $S = \{(e, \xi, f) : \xi \in \Omega, e \in L_{\xi\xi^{-1}}, f \in R_{\xi^{-1}\xi}\}$, and define multiplication $\circ$ in S as follows:

$$(e, \xi, f) \circ (g, \eta, h) = (eu, \xi\eta, vh),$$

where $u \in L_{\xi\eta(\xi\eta)^{-1}}$ and $v \in R_{(\xi\eta)^{-1}\xi\eta}$. Such multiplication $\circ$ is well-defined, since $eL_{\xi\eta(\xi\eta)^{-1}} \subset L_{\xi\eta(\xi\eta)^{-1}}$ and $R_{(\xi\eta)^{-1}\xi\eta}h \subset R_{(\xi\eta)^{-1}\xi\eta}$ and since each of $eL_{\xi\eta(\xi\eta)^{-1}}$ and $R_{(\xi\eta)^{-1}\xi\eta}h$ consists of a single element.

Now, by simple calculation we can easily prove the following lemma:

Lemma 4. *The resulting system $S(\circ)$ is a regular semigroup and the set of idempotents of $S(\circ)$ is a normal band. Hence, $S(\circ)$ is a generalized inverse semigroup.*

We shall call $S(\circ)$ in Lemma 4 the quasi-direct product of L, Ω and R with respect to Γ, and denote it by $Q(L \otimes \Omega \otimes R ; \Gamma)$.[3)]

Now, let S be a generalized inverse semigroup and let B be the normal band consisting of all idempotents of S. Let $B \sim \sum\{B_\gamma : \gamma \in \Gamma\}$ be the structure decomposition of B.

Let us define a relation $\mathfrak{D}$ on S as follows:

(D) $x \mathfrak{D} y$ if and only if $\{x^* : x^* \in S$ and x^* is an inverse of $x\}$ $= \{y^* : y^* \in S$ and y^* is an inverse of $y\}$.

Then $\mathfrak{D}$ is a congruence on S, and the factor semigroup $S/\mathfrak{D}$ of S mod $\mathfrak{D}$ is an inverse semigroup. Further the restriction $\mathfrak{D}_B$ of $\mathfrak{D}$

3) It is easy to see that $Q(L \otimes \Omega \otimes R ; \Gamma)$ is same to the direct product $L \times \Omega \times R$ of L, Ω and R if Γ is a single element, that is, if L, Ω, and R are a left zero semigroup (i.e., left singular semigroup), a group and a right zero semigroup (i.e., right singular semigroup) respectively.

to B gives the structure decomposition of B, and the factor semigroup $B/\mathfrak{D}_B$ $(=\{B_\gamma:\gamma\in\Gamma\})$ of B mod $\mathfrak{D}_B$ is the basic semilattice of $S/\mathfrak{D}$.[4] Next, we also define relations $\mathfrak{R}$, $\mathfrak{L}$ on B as follows:

(R) $e\mathfrak{R}f$ if and only if $ef=f$ and $fe=e$.

(L) $e\mathfrak{L}f$ if and only if $ef=e$ and $fe=f$.

Then $\mathfrak{R}$, $\mathfrak{L}$ are congruences on B satisfying $\mathfrak{R},\mathfrak{L}\leqq\mathfrak{D}_B$, and the factor semigroups $B/\mathfrak{R}$, $B/\mathfrak{L}$ of B mod $\mathfrak{R},\mathfrak{L}$ are a left normal band and a right normal band, having $B/\mathfrak{R}\sim\sum\{B_\gamma/\mathfrak{R}_\gamma:B_\gamma\in B/\mathfrak{D}_B\}$ and $B/\mathfrak{L}\sim\sum\{B_\gamma/\mathfrak{L}_\gamma:B_\gamma\in B/\mathfrak{D}_B\}$ as their structure decompositions respectively, where $\mathfrak{R}_\gamma$ and $\mathfrak{L}_\gamma$ are the restrictions of $\mathfrak{R}$ and $\mathfrak{L}$ to the γ-kernel B_γ of B. Since the basic semilattice of $S/\mathfrak{D}$ is $B/\mathfrak{D}_B$ and since each of the structure semilattices of $B/\mathfrak{R}$ and $B/\mathfrak{L}$ is $B/\mathfrak{D}_B$, we can consider the quasi-direct product $Q(B/\mathfrak{R}\otimes S/\mathfrak{D}\otimes B/\mathfrak{L};\ B/\mathfrak{D}_B)$. Let $\bar{x}$ denote the congruence class containing x mod $\mathfrak{D}$, and let $\tilde{e}$, $\tilde{\tilde{e}}$ denote the congruence classes containing e mod $\mathfrak{R}$, $\mathfrak{L}$ respectively. Now, define a mapping $\psi:S\to Q(B/\mathfrak{R}\otimes S/\mathfrak{D}\otimes B/\mathfrak{L};\ B/\mathfrak{D}_B)$ as follows: $\psi(x)=(\widetilde{xx^*},\bar{x},\widetilde{\widetilde{x^*x}})$, where x^* is an inverse of x. This mapping ψ is well-defined. It can be proved as follows: Let x^* be an inverse of x. Then $\bar{x}^*$ is the inverse of $\bar{x}$ in the inverse semigroup $S/\mathfrak{D}$, and accordingly $\bar{x}\bar{x}^*$, $\bar{x}^*\bar{x}$ are elements of $B/\mathfrak{D}_B$. Let $\bar{x}\bar{x}^*=B_\xi$ and $\bar{x}^*\bar{x}=B_\eta$. Since $\mathfrak{R}\leqq\mathfrak{D}_B$ and $\mathfrak{L}\leqq\mathfrak{D}_B$, it follows that $\widetilde{xx^*}\in B_\xi/\mathfrak{R}_\xi$ and $\widetilde{\widetilde{x^*x}}\in B_\eta/\mathfrak{L}_\eta$. Therefore, $(\widetilde{xx^*},\bar{x},\widetilde{\widetilde{x^*x}})\in Q(B/\mathfrak{R}\otimes S/\mathfrak{D}\otimes B/\mathfrak{L};B/\mathfrak{D}_B)$. Next, let x_1^*, x_2^* be inverses of x. Then $\widetilde{xx_1^*}=\widetilde{xx_2^*}$ and $\widetilde{\widetilde{x_1^*x}}=\widetilde{\widetilde{x_2^*x}}$. Hence $\psi(x)$ is uniquely determined for every x of S. Thus ψ is a mapping of S into $Q(B/\mathfrak{R}\otimes S/\mathfrak{D}\otimes B/\mathfrak{L};\ B/\mathfrak{D}_B)$. It is also easily proved that ψ is an isomorphism of S onto $Q(B/\mathfrak{R}\otimes S/\mathfrak{D}\otimes B/\mathfrak{L};\ B/\mathfrak{D}_B)$.

That is,

Lemma 5. *Let S be a generalized inverse semigroup, and B the normal band consisting of all idempotents of S. Let $B\sim\sum\{B_\gamma:\gamma\in\Gamma\}$ be the structure decomposition of B. Let $\mathfrak{D}$, $\mathfrak{R}$, and $\mathfrak{L}$ be the congruences defined by* (D), (R), *and* (L) *respectively. Let $\mathfrak{D}_B$ be the restriction of $\mathfrak{D}$ to B, and for any γ of Γ $\mathfrak{R}_\gamma$ and $\mathfrak{L}_\gamma$ the restrictions of $\mathfrak{R}$ and $\mathfrak{L}$ to the γ-kernel B_γ of B respectively. Then,*

(1) *$S/\mathfrak{D}$ is an inverse semigroup having $B/\mathfrak{D}_B(=\{B_\gamma:\gamma\in\Gamma\})$ as its basic semilattice, and $B/\mathfrak{R}$ and $B/\mathfrak{L}$ are a left normal band and a right normal band, having $B/\mathfrak{R}\sim\sum\{B_\gamma/\mathfrak{R}_\gamma:B_\gamma\in B/\mathfrak{D}_B\}$ and*

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4) Let M be a subsemigroup of a semigroup S. Let $\mathfrak{P}$ be a relation on S. Define a new relation $\mathfrak{P}_M$ on M as follows: $x\mathfrak{P}_My$ if and only if $x\mathfrak{P}y$, $x,y\in M$. This relation $\mathfrak{P}_M$ is said to be the restriction of $\mathfrak{P}$ to M. It is easy to see that $\mathfrak{P}_M$ is a congruence on M if $\mathfrak{P}$ is a congruence on S.

$B/\mathfrak{L} \sim \sum\{B_\gamma/\mathfrak{L}_\gamma : B_\gamma \in B/\mathfrak{D}_B\}$ *as their structure decompositions; and*

(2) *S is isomorphic to the quasi-direct product* $Q(B/\mathfrak{R} \otimes S/\mathfrak{D} \otimes B/\mathfrak{L};\ B/\mathfrak{D}_B)$.

Summarizing Lemmas 4 and 5, we obtain the following structure theorem for generalized inverse semigroups:

Theorem 2. *A semigroup is a generalized inverse semigroup if and only if it is isomorphic to the quasi-direct product of a left normal band, an inverse semigroup and a right normal band.*

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