27. On Variants of Axiom Systems of Propositional Calculus. I

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In this note, we shall show that any axiom system containing the *BCK*-system of propositional calculus may be effectively changed into a new system which has axioms less than the number of the original axioms. Following certain 'combinatory logicians', we put *B* for *CCqrCCpqCpr*, *C* for *CCpCqrCqCpr*, and *K* for *CpCqp*. This system was given by C. A. Meredith. And Prof. K. Iséki has given the algebraic formulation of the *BCK*-system (see, [5]). For the notations and two rules of inferences, see [4].

Theorem 1. *If F is a thesis in the BCK-system, then CCCpCqpCFvCwv having no occurrences of p, q, v, and w in F implies CpCqp and F.*

The result is obtained by the following proof line.

1 *CCCpCqpCFvCwv.*

1 *p/CpCqp, q/F, v/CpCqp* **C*1 *v/CpCqp, w/F*—2,

2 *CwCpCqp.*

2 *w/CCCpCqpCFvCwv* **C*1—3,

3 *CpCqp.*

1 *v/CCpCqpF, w/CpCqp* **C*2 *p/F, w/CpCqp*—*C*3—*C*3—4,

4 *F.*

Hence thesis 1 implies *CpCqp* and *F*, which completes the proof of Theorem 1.

Theorem 2. *If F is a formula in the BCK-system, then this system implies CCCpCqpCFvCwv, where p, q, v, and w do not contain in F.*

Proof. The axioms of the *BCK*-system are given by the following:

1′ *CCqrCCpqCpr,*

2′ *CCpCqrCqCpr,*

3′ *CpCqp.*

It is well known that these axioms imply (see, [1]),

4′ *CCpqCCqrCpr,*

5′ *CPCCpqq.*

Then we have the following theses:

5′ *p/F, q/v* **CF*—1,

1 *CCFvv,*

1′ q/CFv, r/v, $p/CpCqp$ *$C1$—2,

2 $CCCpCqpCFvCCpCqpv$.

2′ $p/CCpCqpCFv$, $q/CpCqp$, r/v *$C2$—$C3'$—3,

3 $CCCpCqpCFvv$.

4′ $p/CCpCqpCFv$, q/v, r/Cwv *$C3$—$C2'$ p/v, q/w—4,

4 $CCCpCqpCFvCwv$.

We have $CCCpCqpCFvCwv$, therefore the proof is complete.

By the theorems 1 and 2, we can make many new axiom systems of propositional calculi as follows.

The classical propositional calculus contains axioms of the BCK-system, hence, each of the following 1)—15) gives axiom systems of the classical two valued propositional calculus (see, [1], [6], and [7]).

1) $CCCpCqpCCrCstCCrsCrtvCwv$, $CCpqCNqNp$, $CNNpp$, $CpNNp$.

2) $CCCpCqpCCCrsCNsNrvCwv$, $CCpCqrCCpqCpr$, $CNNpp$, $CpNNp$.

3) $CCCpCqpCCNNrrvCwv$, $CCpCqrCCpqCpr$, $CCpqCNqNp$, $CpNNp$.

4) $CCCpCqpCCrNNrvCwv$, $CCpCqrCCpqCpr$, $CCpqCNqNp$, $CNNpp$.

5) $CCCpCqpCCCrsCCstCrtvCwv$, $CCpCqrCqCpr$, $CNNpp$, $CCpNpNp$, $CCpNqCqNp$.

6) $CCCpCqpCCCrCstCsCrtvCwv$, $CCpqCCqrCpr$, $CNNpp$, $CCpNpNp$, $CCpNqCqNp$.

7) $CCCpCqpCCNNrrvCwv$, $CCpqCCqrCpr$, $CCpCqrCqCpr$, $CCpNpNp$, $CCpNqCqNp$.

8) $CCCpCqpCCCrNrNrvCwv$, $CCpqCCqrCpr$, $CCpCqrCqCpr$, $CNNpp$, $CCpNqCqNp$.

9) $CCCpCqpCCCrNsCsNrvCwv$, $CCpqCCqrCpr$, $CCpCqrCqCpr$, $CNNpp$, $CCpNpNp$.

10) $CCCpCqpCCCrCstCsCrtvCwv$, $CCqrCCpqCpr$, $CpCNpq$, $CCpqCCNpqq$.

11) $CCCpCqpCCCrsCCtrCtsvCwv$, $CCpCqrCqCpr$, $CpCNpq$, $CCpqCCNpqq$.

12) $CCCpCqpCCrCNrsvCwv$, $CCpCqrCqCpr$, $CCqrCCpqCpr$, $CCpqCCNpqq$.

13) $CCCpCqpCCCrsCCNrssvCwv$, $CCpCqrCqCpr$, $CCqrCCpqCpr$, $CpCNpq$.

14) $CCCpCqpCCCrCstCCrsCrtvCwv$, $CCNpNqCqp$.

15) $CCCpCqpCCCNrNsCsrvCwv$, $CCpCqrCCpqCpr$.

Similarly, for example, we have the following axiom systems of implicational calculus (see, [3]).

1) *CCCpCqpCCCrsCCstCrtvCwv*, *CCCpqpp*.
2) *CCCpCqpCCCCrsrrvCwv*, *CCpqCCqrCpr*.
3) *CCCpCqpCCCCCrstuCCsuCruvCwv*.
4) *CCCpCqpCCCCCrstsCCsuCruvCwv*.

For the positive implicational calculus, we have the following axiom systems 1)—7).

1) *CCCpCqpCCCrCrsCrsvCwv*, *CCqrCCpqCpr*, *CCpCqrCqCpr*.
2) *CCCpCqpCCCrsCCtrCtsvCwv*, *CCpCpqCpq*, *CCpCqrCqCpr*.
3) *CCCpCqpCCCrCstCsCrtvCwv*, *CCpCpqCpq*, *CCqrCCpqCpr*.
4) *CCCpCqpCCCrCrsCrsvCwv*, *CCpqCCqrCpr*.
5) *CCCpCqpCCCrsCCstCrtvCwv*, *CCpCpqCpq*.
6) *CCCpCqpCCCrCstCCrsCrtvCwv*.
7) *CCCpCqpCCCrsCCrCstCrtvCwv*.

Further we can mention the following variants of the *BCK*-system (see, [2]).

1) *CCCpCqpCCCrsCCtrCtsvCwv*, *CpCCpqq*.
2) *CCCpCqpCCrCCrssvCwv*, *CCqrCCpqCpr*.
3) *CCCpCqpCCCrsCCstCrtvCwv*, *CpCCpqq*.
4) *CCCpCqpCCrCCrssvCwv*, *CCpqCCqrCpr*.
5) *CCCpCqpCCCrsCCtrCtsvCwv*, *CCpCqrCqCpr*.
6) *CCCpCqpCCCrCstCsCrtvCwv*, *CCqrCCpqCpr*.
7) *CCCpCqpCCCrCstCCusCuCrtvCwv*.
8) *CCCpCqpCCCrsCCtCusCrCtuvCwv*.
9) *CCCpCqpCCCCCrstuCCstCruvCwv*.
10) *CCCpCqpCCCrsCCCCtrsuCtuvCwv*.

References

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