25. Axiom Systems of B-algebra. V

By Kashiko TANAKA

(Comm. by Kinjirô KUNUGI, M.J.A., Feb. 12, 1966)

In this note, we shall give a characterization of B-algebra by an algebraic formulation of an axiom system of propositional calculus by B. Sobociński.

We introduce a binary operation $*$ and an unary operation $\sim$ on X. Consider an abstract algebra $\langle X, 0, *, \sim\rangle$ satisfying the following conditions:

B 1 $x*y\leqslant x$,
B 2 $(x*y)*(z*y)\leqslant(x*z)*y$,
B 3 $x*y\leqslant\sim y*\sim x$,
B 4 $0\leqslant x$,
B 5 $x\leqslant y$ and $y\leqslant x$ imply $x=y$,
B 6 $x*y=0$ if and only if $x\leqslant y$

(For details, see [1].)

The algebra $\langle X, 0, *, \sim\rangle$ is called a B-algebra.

By the same idea, we can formulate an axiom system by B. Sobociński as follows:

S 1 $x*y\leqslant\sim y$,
S 2 $(x*y)*z\leqslant x$,
S 3 $(x*(y*z))*(x*y)\leqslant x*\sim z$,

and B 4, B 5, B 6.

In the case of propositional calculus, it is called the (S_1)-system.

We shall prove that two axiom systems are equivalent, therefore any B-algebra, hence a Boolean algebra is characterized by the above axiom S 1–3 and B 4–6.

First we shall show that the axioms of B-algebra are derived from the axioms of (S_1)-system.

In axiom S 3, put $x=(x*y)*z$, $y=x$, then we have $((x*y)*z)*(x*z))*(((x*y)*z)*x)\leqslant((x*y)*z)*\sim z$. In axiom S 1, put $x=x*y$, $y=z$, then we have $((x*y)*z)*\sim z=0$ by B 6. Hence $((x*y)*z)*(x*z)\leqslant((x*y)*z)*x$. The right side is equal to 0 by S 2, then

(1) $(x*y)*z\leqslant x*z$.

In (1), put $x=x*y$, $y=z$, and $z=\sim y$, then by S 1 and B 6, we have $((x*y)*z)*\sim y=0$. Hence by B 6,

(2) $(x*y)*z\leqslant\sim y$.

In S 3, put $x=(x*y)*z$, $y=x*z$, and $z=y$, then the right side is equal to 0 by (2) and the second term of left side is to 0 by (1). Therefore we have

(3) $(x*y)*z \leqslant (x*z)*y$.

In (3), put $y=z$, $z=y$, then $(x*z)*y \leqslant (x*y)*z$. Therefore by $B\,5$ we have

(4) $(x*y)*z=(x*z)*y$.

In $S\,3$, put $x=x*z$, then we have $((x*z)*(y*z))*((x*z)*y) \leqslant (x*z)*\sim z=0$ by $S\,1$. Therefore $(x*z)*(y*z) \leqslant (x*z)*y=(x*y)*z$ by (4). Hence we have

(5) $(x*z)*(y*z) \leqslant (x*y)*z$.

In (5), put $x*y=0$ and $y*z=0$, then we have $x*z=0$ by $B\,4$ and $B\,6$. Hence by $B\,6$ we have the following important.

(6) $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$.

In (4), put $x=x*y$, $y=z$, and $z=x$, then we have $((x*y)*z)*x=((x*y)*x)*z=0$ by $S\,2$. Since z is arbitrary, $(x*y) \leqslant x$.

(7) $x*y \leqslant x$.

Let us put $y=x$ in (7), and we use (4), then $(x*x)*z=(x*z)*x=0$. The element z is arbitrary, hence

(8) $x*x=0$.

In (1), put $z=y$, $y=z$, then we have $(x*z)*y \leqslant x*y$. By (4) and (5), $(x*z)*(y*z) \leqslant (x*y)*z=(x*z)*y$. Hence we have $(x*z)*y \leqslant (x*y)$ and $(x*z)*(y*z) \leqslant (x*z)*y$. Therefore, by (6), we have

(9) $(x*z)*(y*z) \leqslant x*y$.

In (4), put $y=\sim y$, $z=y$, then $(x*\sim y)*y \leqslant (x*y)*\sim y=0$ by $S\,1$. Hence we have

(10) $x*\sim y \leqslant y$.

In $S\,3$, put $x=x*\sim\sim x$, $y=x$, $z=x$, then we have $((x*\sim\sim x)*(x*x))*((x*\sim\sim x)*x) \leqslant (x*\sim\sim x)*\sim x$. The right side is equal to 0 by (10). Further we have $(x*\sim\sim x)*x=0$ by putting $y=\sim\sim x$ in (7), $x*x$ is equal to 0 by (8). Therefore we have

(11) $x*\sim\sim x=0$.

In $S\,3$, put $x=\sim x*y$, $z=x$, $y=\sim y$, then we have $((\sim x*y)*(\sim y*x))*((\sim x*y)*\sim y) \leqslant (\sim x*y)*\sim x=0$. From (7) and $S\,1$, we have $(\sim x*y)*\sim x=0$ and $(\sim x*y)*\sim y=0$. Therefore

(12) $\sim x*y \leqslant \sim y*x$.

In (12), put $x=\sim x$, $y=x$, then by (8) we have

(13) $\sim\sim x*x=0$.

Formulas (11), (13), and $B\,5$ imply

(14) $\sim\sim x=x$.

In axiom $S\,3$, put $x=x*y$, $y=\sim y$, and $z=\sim x$, then we have $((x*y)*(\sim y*\sim x))*((x*y)*\sim y) \leqslant (x*y)*\sim\sim x=(x*y)*x=0$ by $S\,1$ and (14). The second term of left side is equal to 0 by $S\,1$. Hence

(15) $x*y \leqslant \sim y*\sim x$.

Hence formulas (5), (7), and (15) are $B\,2$, $B\,1$, $B\,3$ respectively.

Next we shall show that the axioms of (S_1)-system are derived from the axioms of B-algebra.

Suppose $x*z=0$ in axiom $B\,2$, the right side is equal to 0 by $B\,4$. Further suppose $z*y=0$ in $B\,2$, then we have the following two results,

Lemma 1. *If* $x*z=0$, *then* $(x*y)*(z*y)=0$, *i.e.* $x\leqslant z$ *imply* $x*y\leqslant z*y$.

Lemma 2. *If* $x*z=0$, $z*y=0$, *then* $x*y=0$, *i.e.* $x\leqslant z$, $z\leqslant y$ *imply* $x\leqslant y$.

In his paper (see [1]), K. Iséki has proved that the axiom system of B-algebra implies $x*x=0$, $x=\sim\sim x$. Therefore we use these result;

(1′) $x*x=0$,

(2′) $x=\sim\sim x$.

In $B\,3$, put $x=\sim y$, $y=\sim x$, then $\sim y*\sim x\leqslant\sim\sim x*\sim\sim y=x*y$ by (2′). Hence we have $\sim y*\sim x\leqslant x*y$. Then by $B\,3$ and $B\,5$,

(3′) $x*y=\sim y*\sim x$.

In axiom $B\,1$, put $x=x*z$, then $(x*z)*y\leqslant x*z$. On the other hand we have $(x*y)*(z*y)\leqslant(x*z)*y$. Hence by Lemma 2 we have,

(4′) $(x*y)*(z*y)\leqslant x*z$.

In $B\,2$, put $x=(x*y)*z$, $y=(x*y)*(z*y)$, $z=(x*y)*z$, then $((x*y)*z)*((x*y)*(z*y))*(((z*y)*z)*((x*y)*(z*y)))\leqslant(((x*y)*z)*((z*y)*z))*((x*y)*(z*y))$. The right side is equal to 0, because it is obtain by substituting $x*y$ for x, z for y, and $z*y$ for z in (4′). Therefore $((x*y)*z)*((x*y)*(z*y))\leqslant((z*y)*z)*((x*y)*(z*y))$. The right side equal to 0 by $B\,1$, $B\,4$. Hence,

(5′) $(x*y)*z\leqslant(x*y)*(z*y)$.

By (5′), $B\,2$ and Lemma 2 we have,

(6′) $(x*y)*z\leqslant(x*z)*y$.

In (6′), put $y=z$, $z=y$, then $(x*z)*y\leqslant(x*y)*z$. By (6′) and $B\,5$ we have,

(7′) $(x*z)*y=(x*y)*z$.

Let us put $x=y*z$ in (1′), then $(y*z)*(y*z)=0$. In (7′), put $x=y$, $y=y*z$, then $(y*z)*(y*z)=(y*(y*z))*z=0$. Hence $y*(y*z)\leqslant z$, and by Lemma 1 we have,

(8′) $(y*(y*z))*\sim x\leqslant z*\sim x$.

In axiom $B\,2$, put $x=\sim(y*z)$, $y=\sim x$, and $z=\sim y$, then we have $(\sim(y*z)*\sim x)*(\sim y*\sim x)\leqslant(\sim(y*z)*\sim y)*\sim x=(y*(y*z))*\sim x\leqslant z*\sim x$ by (8′), (3′), and Lemma 2. Hence,

(9′) $(\sim(y*z)*\sim x)*(\sim y*\sim x)\leqslant z*\sim x$.

In axiom $B\,3$, put $x=\sim y$ and $y=\sim x$, then $\sim y*\sim x\leqslant\sim\sim x*\sim\sim y=x*y$ by (2′). Hence we have $\sim y*\sim x\leqslant x*y$ and $x*y\leqslant\sim y*\sim x$ by axiom $B\,3$. Therefore we have $x*y=\sim y*\sim x$ by axiom $B\,5$.

(10′) $x*y=\sim y*\sim x$.

By (10′), (2′) we have $\sim(y*z)*\sim x=x*(y*z)$, $\sim y*\sim x=x*y$, and $z*\sim x=\sim\sim x*\sim z=x*\sim z$. Therefore by (9′) we have,

(11′) $(x*(y*z))*(x*y)\leqslant x*\sim z$.

By $B\,1$ and Lemma 1, we have $(x*y)*z\leqslant x*z$. On the other hand $x*z\leqslant x$ by $B\,1$. By Lemma 2 we have,

(12′) $(x*y)*z\leqslant x$.

In axiom $B\,1$, put $x=\sim y$, $y=\sim x$, then we have $\sim y*\sim x\leqslant\sim y$. On the other hand $x*y\leqslant\sim y*\sim x$ by $B\,3$. Therefore by Lemma 2, we have,

(13′) $x*y\leqslant\sim y$.

Hence formulas (11′), (12′), and (13′) are $S\,3$, $S\,2$, and $S\,1$ respectively.

Therefore we can conclude that the axiom system of B-algebra is equivalent to the (S_1)-system.

Reference

[1] K. Iséki: Algebraic formulations of propositional calculi. Proc. Japan Acad., **41**, 803-807 (1965).