57. On the Strong (L) Summability of the Derived Fourier Series

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1. In a recent paper, Borwein [1] has constructed a new method of summability for an infinite sequence $\{s_n\}$. He defines a sequence $\{s_n\}$ to be summable by the logarithmic method of summability or summable (L) to the sum s if, for x in the interval $(0, 1)$,

$$(1.1) \qquad \lim_{x\to 1-0} \frac{1}{\log(1-x)} \sum_{n=1}^{\infty} \frac{s_n}{n} x^n = s.$$

It is known [3] that this method includes the Abel method. Recently K. Ishiguro [4] proved that if $\{s_n\}$ is summable by Riesz logarithmic mean of order one, it is also summable (L) to the same sum, but the converse is not true.

A series $c_0+c_1+c_2+\cdots$ is said to be strongly summable $(c, 1)$ or summable $[c, 1]$ to the sum s, if

$$(1.2) \qquad \sum_{\nu=0}^{n} |s_\nu - s| = o(n), \qquad \text{as } n\to\infty,$$

s_ν being the sum of the first $(\nu+1)$ terms of the series. The series is said to be strongly summable by Riesz logarithmic mean of order one or summable $[R, \log n, 1]$ to the sum s, if

$$(1.3) \qquad \sum_{\nu=0}^{n} \frac{|s_\nu - s|}{\nu} = o(\log n), \qquad \text{as } n\to\infty.$$

We define an analogue for strong summability of (L) summability method as follows:

Definition. A series $\sum_{n=0}^{\infty} c_n$ with the sequence of partial sum $\{s_n\}$ is said to be summable by strong (L) summability to the sum s if

$$(1.4) \qquad \sum_{\nu=1}^{\infty} \frac{x^\nu |s_\nu - s|}{\nu} = o\{\log(1-x)\}, \qquad \text{as } x\to 1$$

for x in the open interval $(0, 1)$.

2. Suppose that the function $f(t)$ is Lebesgue integrable over the interval $(0, 2\pi)$ and periodic with period 2π. Let the Fourier series associated with function $f(t)$ be

$$(2.1) \qquad \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{1}^{\infty} A_n(t).$$

The series

$$(2.2) \qquad \sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) = \sum_{1}^{\infty} nB_n(t),$$

which is obtained by differentiating the series (2.1) term by term, is called the derived Fourier series of $f(t)$.

We write

$$\psi(t) = \frac{1}{2}\{f(\xi+t) - f(\xi-t)\}$$

$$g(t) = \frac{1}{2}\{\psi(t) - tf'(\xi)\},$$

where $f'(\xi)$ denotes the first generalised differential coefficients of $f(t)$ at $t=\xi$.

For the first time Prasad and Singh [5] gave criteria for the strong summability of the derived Fourier series. They proved the following:

Theorem A. *If $f(t)$ be a continuous function of bounded variation and if for some value of ξ and for some $\varepsilon > 0$*

$$(2.3) \qquad \int_0^t |dg(u)| = o\left\{\frac{t}{(\log 1/t)^{1+\varepsilon}}\right\}, \qquad \textit{as } t \to 0,$$

then

$$(2.4) \qquad \sum_{\nu=1}^{n} |s_\nu(\xi) - f'(\xi)| = o(n).$$

Further Chow [3] has localised and generalised the above theorem and proved the following

Theorem B. *If*

$$(2.5) \qquad \sum_{\nu=1}^{n} \nu\, |B_\nu(\xi)| = o(n)$$

(2.6) *the function $\dfrac{\psi(t)}{t}$ is of bounded variation in a neighbourhood of $t=0$, then* (2.4) *holds.*

In the subsquent section we shall investigate the strong (L) summability of the derived Fourier series. In fact we prove:

Theorem C. *If*

$$(2.7) \qquad \sum_{\nu=1}^{n} x^\nu |B_\nu(\xi)| = o\{\log(1-x)\}, \qquad x \to 1 \text{ in } 0 < x < 1,$$

and

$$(2.8) \qquad \varepsilon(t) = \int_t^\pi \frac{|dg(u)|}{u} = o\left(\log \frac{1}{t}\right)$$

then

$$(2.9) \qquad \sum_{\nu=1}^{\infty} \frac{x^\nu |s_\nu(\xi) - f'(\xi)|}{\nu} = o\{\log(1-x)\}.$$

It should be noted here that (2.8) implies that

$$(2.10) \qquad \int_0^t |dg(u)| = o\left(t \log \frac{1}{t}\right).$$

3. Proof of theorem C. We have

$$s_n(\xi)=\frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{d}{d\xi}\frac{\sin\left(n+\frac{1}{2}\right)(\xi-u)}{\sin\frac{1}{2}(\xi-u)}\right\}f(u)du$$

$$=-\frac{1}{2\pi}\int_0^{2\pi}f(u)\left\{\frac{d}{du}\frac{\sin\left(n+\frac{1}{2}\right)(\xi-u)}{\sin\frac{1}{2}(\xi-u)}\right\}du$$

$$=-\frac{1}{2\pi}\int_0^{2\pi}\{f(\xi+t)-f(\xi-t)\}\left\{\frac{d}{dt}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin t/2}\right\}dt. \tag{3.1}$$

Integrating by parts the right hand side of (3.1), we obtain

$$s_n(\xi)=\frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin t/2}d\{f(\xi+t)-f(\xi-t)\}$$

$$=\frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dg(t)+f'(\xi).$$

Now

$$\sum_{\nu=1}^{\infty}\frac{x^\nu\,|\,s_n(\xi)-f'(\xi)\,|}{\nu}=\frac{1}{2\pi}\int_0^{\pi}\left(\sum_{\nu=1}^{\infty}\frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin t/2}\frac{x^\nu}{\nu}\right)dg(t)$$

$$=\frac{1}{2\pi}\int_0^{\pi}\frac{dg(t)}{\tan t/2}\left(\sum_{\nu=1}^{\infty}\frac{\sin\nu t}{\nu}x^\nu\right)$$

$$+\frac{1}{2\pi}\int_0^{\pi}dg(t)\left(\sum_{\nu=1}^{\infty}\frac{\cos\nu t}{\nu}x^\nu\right)$$

$$=I_1+I_2. \tag{3.2}$$

Also

$$\frac{1}{2\pi}\int_0^{\pi}\cos\nu t dg(t)=\frac{1}{\pi}\int_0^{\pi}\cos\nu t d\psi(t)+o(1)$$

$$=\left[\frac{\cos\nu t}{\pi}\psi(t)\right]_0^{\pi}+\frac{\nu}{\pi}\int_0^{\pi}\sin\nu t\psi(t)dt+o(1)$$

$$=\frac{\nu}{\pi}\int_{-\pi}^{\pi}f(u)\sin\nu(u-\xi)du+o(1)$$

$$=\nu(b_\nu\cos\nu\xi-a_\nu\sin\nu\xi)+o(1)$$

$$=\nu B_\nu(\xi)+o(1),$$

so that

$$|\,I_2\,|=\left|\sum_1^{\infty}\frac{x^\nu}{\nu}\frac{1}{2\pi}\int_0^{\pi}dg(t)\cos\nu t\right|$$

$$=\left|\sum_{\nu=1}^{\infty}\frac{x^{\nu}}{\nu}\cdot\nu B_{\nu}(\xi)\right|+o\{\log(1-x)\}$$

$$=\left|\sum_{\nu=1}^{\infty}x^{\nu}B_{\nu}(\xi)\right|+o\{\log(1-x)\}$$

$$=o\{\log(1-x)\},\quad \text{by (2.7)}. \tag{3.3}$$

Further

$$I_1=\frac{1}{2\pi}\left\{\int_0^{1-x}+\int_{1-x}^{\pi}\right\}\left(\tan^{-1}\frac{x\sin t}{1-x\cos t}\right)\frac{dg(t)}{\tan t/2}$$

$$=\frac{1}{2\pi}(I_{1,1}+I_{1,2}),\qquad \text{say}.$$

It is easy to see that

$$\left|\frac{1}{\tan t/2}\tan^{-1}\frac{x\sin t}{1-x\cos t}\right|=O\left(\frac{x}{1-x}\right),\qquad \text{for } 0<t\leq 1-x, \tag{3.4}$$

and

$$\left|\tan^{-1}\frac{x\sin t}{1-x\cos t}\right|=O(1),\qquad \text{for } 1-x<t\leq\pi. \tag{3.5}$$

Using (3.4) and (2.10), we have,

$$|I_{1,1}|=O\left(\frac{x}{1-x}\right)\int_0^{1-x}|dg(t)|$$

$$=o\{\log(1-x)\}. \tag{3.6}$$

With the help of (3.5) and (2.8), we write,

$$|I_{1,2}|=O(1)\int_{1-x}^{\pi}\frac{|dg(t)|}{t}$$

$$=o\{\log(1-x)\}. \tag{3.7}$$

Collecting (3.3), (3.6), and (3.7), the proof of theorem is complete.

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