77. On the So-called Fundamental Theorem of Integration

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Hobson [1] called *fundamental theorem of the integration* the following property of the integral that *the indefinite Denjoy integral (in the restricted sense) has finite differential coefficient, equal to the integrand, almost everywhere in the interval of integration.*

Concerning this theorem, we shall give a more general condition. For this purpose, consider in the first place A-integral introduced by E. C. Titchmarsh [2] and A. Kolgomoroff [3] which enables us to integrate every function conjugate with a summable function. It seems that A-integral is not only too general to have the unicity but also too special to integrate every $\mathfrak{D}$-integrable [4] function and the function $\frac{1}{x}$.

On the other hand (E.R.)-integral was defined by K. Kunugi [5]. Although it is equivalent to A-integral [6], the generalized (E.R.)-integral, named (E.R. ν)-integrals by H. Okano [7], enables us to integrate the function $\frac{1}{x}$, and also enables us, for every $\mathfrak{D}$-integrable function $f(x)$, to find an (E.R. ν)-integral which gives the indefinite (E.R. ν)-integral of $f(x)$ identical with the indefinite $\mathfrak{D}$-integral of $f(x)$ [8].

Let us begin to consider the following condition (Q*) of Cauchy sequence which may be regarded as a modification of the condition (Q) introduced by H. Okano;

(Q*) i) A_n is a non-decreasing sequence of closed sets,

ii) $\sum_{i=1}^{\infty}\left|\int_{I_n^i} f_m(t)dt\right| < \varepsilon_n$ for $m \geqq n$,

where $\{I_n^i\}$ be the sequence of intervals contiguous[1] to A_n.

Theorem 1.[2] *If a function $f(x)$ has a Cauchy sequence $(V(\mathcal{E}_n, A_n; f_n)) \in \mathfrak{C}(f; \nu)$ which satisfies the condition (Q^*), then $F(x)=(E.R.\nu)\int_{-\infty}^{x} f(t)dt$ exists and is finite for every point $x \in A_n$, and we have*

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1) We say that a closed interval I is contiguous to a closed set F when the interior of I is a connected compoment of CF.

2) For the notations see [7].

$$(E.R.\nu)\int_{-\infty}^{x} f(t)dt=\int_{A_n\cap(-\infty,x)} f(t)dt+\sum_{\{i:\, I_n^i\subset(-\infty,x)\}}(E.R.\nu)\int_{I_n^i} f(t)dt$$

for every $x\in A_n$ *and*

$$\sum_i\left|(E.R.\nu)\int_{I_n^i} f(t)dt\right|<\varepsilon_n$$

where $\{I_n^i\}$ *be the sequence of intervals contiguous to* A_n.

Proof. Let $x\in A_n$, then we have

$$\left|\int_{-\infty}^{x} f_n(t)dt-\int_{-\infty}^{x} f_m(t)dt\right|$$

$$\leqq\int_{A_n}|f_n(t)-f_m(t)|\,dt+\sum_{i=1}^{\infty}\left|\int_{I_n^i} f_n(t)dt\right|+\sum_{i=1}^{\infty}\left|\int_{I_n^i} f_m(t)dt\right|\leqq 3\varepsilon_n$$

for every integer $m\geqq n$. Hence we have

$$\left|(E.R.\nu)\int_{-\infty}^{x} f(t)dt-\int_{-\infty}^{x} f_n(t)dt\right|\leqq 3\varepsilon_n. \tag{1}$$

Next, let $x\in A_n$ and let $\{I_n^i\}$ be the sequence of intervals contiguous to A_n, then we have, for every pair m, m' of integers ($n\leqq m\leqq m'$),

$$\sum_i\left|\int_{I_n^i} f_m(t)dt-\int_{I_n^i} f_{m'}(t)dt\right|$$

$$\leqq\sum_i\left|\int_{A_m\cap I_n^i}(f_m(t)-f_{m'}(t))dt+\sum_{\{j:\, I_m^j\subset I_n^i\}}\int_{I_m^j}(f_m(t)-f_{m'}(t))dt\right|$$

$$\leqq\int_{A_m}|f_m(t)-f_{m'}(t)|\,dt+\sum_{j=1}^{\infty}\left|\int_{I_m^j} f_m(t)dt\right|+\sum_{j=1}^{\infty}\left|\int_{I_m^j} f_{m'}(t)dt\right| \tag{2}$$

$$\leqq 3\varepsilon_m.$$

Hence, we have, for every integer $m\geqq n$,

$$\sum_i\left|(E.R.\nu)\int_{I_n^i} f(t)dt-\int_{I_n^i} f_m(t)dt\right|\leqq 3\varepsilon_m. \tag{3}$$

It follows from the relations (1) and (2), that

$$\left|(E.R.\nu)\int_{-\infty}^{x} f(t)dt-\int_{A_n\cap(-\infty,x)} f(t)dt-\sum_i(E.R.\nu)\int_{I_n^i} f(t)dt\right|$$

$$\leqq\left|(E.R.\nu)\int_{-\infty}^{x} f(t)dt-\int_{-\infty}^{x} f_m(t)dt\right|+\int_{A_n}|f(t)-f_m(t)|\,dt$$

$$+\sum_i\left|(E.R.\nu)\int_{I_n^i} f(t)dt-\int_{I_n^i} f_m(t)dt\right|\leqq 7\varepsilon_m$$

for every point $x\in A_n$ and every integer $m\geqq n$. Hence we have

$$(E.R.\nu)\int_{-\infty}^{x} f(t)dt=\int_{A_n\cap(-\infty,x)} f(t)dt+\sum_{\{i:\, I_n^i\subset(-\infty,x)\}}(E.R.\nu)\int_{I_n^i} f(t)dt \tag{4}$$

for every point $x\in A_n$.

It follows from the property (Q*) and the relation (3) that

$$\sum_i\left|(E.R.\nu)\int_{I_n^i} f(t)dt\right|$$

$$\leqq\sum_i\left\{\left|(E.R.\nu)\int_{I_n^i} f(t)dt-\int_{I_n^i} f_m(t)dt\right|+\left|\int_{I_n^i} f_m(t)dt\right|\right\}\leqq 3\varepsilon_m+\varepsilon_n$$

for every integer $m\geqq n$. Hence we have

$$\sum_i\left|(E.R.\nu)\int_{I_n^i} f(t)dt\right|\leqq\varepsilon_n \tag{5}$$

for every integer n. This completes the proof.

Theorem 2. *If a function $f(x)$ has a Cauchy sequence $(V(\varepsilon_n)A_n; f_n))\in\mathfrak{C}(f;\nu)$ which satisfies the condition (Q*), then $F(x)=(E.R.\nu)\int_{-\infty}^{x} f(t)dt$ is AC on every A_n and has the approximate derivative $F'_{ap}(x)=f(x)$ almost all points x of the whole interval of integration.*

Proof. First, let n be a fixed integer and let $\{I_n^i\}$ be the sequence of intervals contiguous to A_n, then, being $f(x)$ summable on A_n, there exists $\eta'=\eta'(\varepsilon, n)>0$ for every $\varepsilon>0$ such that

$$\text{mes}(E)<\eta' \quad \text{implies} \int_{E\cap A_n}|f(x)|\,dx<\frac{\varepsilon}{2}. \tag{6}$$

Writing $F(I)=(E.R.\nu)\int_I f(t)dt$, there exists, for every $\varepsilon>0$, an integer $k=k(\varepsilon, n)$ such that

$$\sum_{i=k+1}^{\infty}|F(I_n^i)|<\frac{\varepsilon}{2}.$$

Let $\{J_j\}$ be a sequence of intervals whose end points belong to A_n, then the inequality $\sum_j|J_j|<\eta=\min(\eta', |I_n^1|, |I_n^2|, \cdots, |I_n^k|)$ implies

$$\sum_j|F(J_j)|=\sum_j\left|\int_{A_n\cap J_j} f(t)+\sum_{\{i:\, I_n^i\subset J_j\}} F(I_n^i)\right|$$
$$\leqq\int_{A_n\cap(\cup J_j)}|f(t)|\,dt+\sum_{i=k+1}^{\infty}|F(I_n^i)|<\varepsilon.$$

Hence $F(x)$ is AC on A_n.

Next, let

$$g_n(x)=\begin{cases} f(x) & \text{for } x\in A_n \\ F(I_n^i)/|I_n^i| & \text{for } x\in I_n^i, \end{cases}$$

then $g_n(x)$ is summable. Writing $G_n(x)=\int_{c_n}^{x} g_n(t)dt+(E.R.\nu)\int_{-\infty}^{c_n} f(t)dt$ where c_n is the greatest lower bound of A_n, we have $G_n(x)=F(x)$ and $G'_n(x)=g_n(x)$ for almost all points x of A_n. Hence, we have $f(x)=g_n(x)=G'_n(x)=F'_{ap}(x)$ for almost all points x of A_n and $F'_{ap}(x)=f(x)$ almost everywhere. This completes the proof.

Corollary. (1) *If the function $f(x)$ has a Cauchy sequence $(V(\varepsilon_n, A_n; f_n))\in\mathfrak{C}(f;\nu)$ which satisfies in addition to the condition (Q^*), the conditions* iii) $(a, b)-\bigcup_{n=1}^{\infty}A_n=\phi$, iv) $\left|\int_a^x(f_n(t)-f_m(t))dt\right|<\varepsilon_n+\varepsilon_m$ *for every point of the interval (a, b), then $f(x)$ is $\mathfrak{D}$-integrable on (a, b) and we have*

$$(\mathfrak{D})\int_a^x f(t)dt=(E.R.\nu)\int_a^x f(t)dt$$

for every point x of the interval (a, b).

(2) *Conversely, if a function $f(x)$ is $\mathfrak{D}$-integrable on the interval (a, b), there exists a $(E.R.\nu)$-integral and a Cauchy sequence $(V(\varepsilon_n, A_n; f_n)) \in \mathfrak{C}(f; \nu)$ which satisfies the conditions* iii) *and* iv) *in addition to (Q^*).*

Proof. It follows from iv) and theorem 1 that there exists finite $F(x)=(E.R.\nu)\int_a^x f(t)dt$ for every point x of (a, b). And it follows from iv) that $F(x)$ is continuous on the interval (a, b). From this and theorem 1 $F(x)$ is ACG on the interval (a, b) and we have $F'_{ap}(x)=f(x)$ almost all points x of (a, b). Hence we have

$$(\mathfrak{D})\int_a^x f(t)dt=F(x)=(E.R.\nu)\int_a^x f(t)dt.$$

The converse is clear by the corollary which we have given in the paper "On indifinite (E.R)-integrals. II."

Example 1. Let

$$f(x)=\begin{cases}\dfrac{1}{x} & \text{for } x\in[-1,1]\\ 0 & \text{otherwise,}\end{cases}$$

$$\nu(E)=\int_E \frac{e^{-|x|}}{x^2}dx,$$

$$\varepsilon_n=\frac{2}{n}$$

$$A_n=\left(-\infty, -\frac{1}{n}\right]\cup\left[\frac{1}{n}, +\infty\right)$$

$$f_n(x)=\begin{cases}f(x) & \text{for } x\in A_n\\ 0 & \text{otherwise,}\end{cases}$$

then the Cauchy sequence $(V(\varepsilon_n, A_n; f_n)) \in \mathfrak{C}(f; \nu)$ satisfies the condition (Q^*). And we have $F(x)=(E.R.\nu)\int_{-\infty}^x f(t)dt=\log|x|, F'_{ap}(x)=F'(x)=\dfrac{1}{x}$ for every point x of $(-1, 1)-\{0\}$.

Example 2. Let us begin to define a sequence $\{H_n\}$ of closed sets in $I_0=[0, 1]$. Let $H(I)$ be the Harnack set in $I=[a, b]$ whose measure is equal to $\dfrac{|I|}{2}$, and let $H^+(I)$ and $H^-(I)$ be the set of all points x of $H(I)$ such that $x\leq\dfrac{a+b}{2}$ and $x>\dfrac{a+b}{2}$ respectively. Let $H_1=H(I_0)$ and $H_{n+1}=\bigcup_{i=1}^{\infty} H(I_n^i)\cup H_n$, where $\{I_n^i\}$ be the sequence of the intervals in $[0, 1]$ contiguous to H_n. Then $\{H_n\}$ is the non-decreasing sequence of closed sets.

Let

$$f(x)=\begin{cases} \pm\dfrac{2^n}{n} & \text{for } x\in H^{\pm}(I_n^i) \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu(E)=\sum_{n=1}^{\infty} 2^{-n}\,\mathrm{mes}\,(E\cap H_n),$$

$$A_n=H_n\cup(-\infty,\,0]\cup[1,\,\infty),$$

$$\varepsilon_n=\frac{1}{n},$$

$$f_n=\begin{cases} f(x) & \text{for } x\in A_n \\ 0 & \text{otherwise,} \end{cases}$$

then the Cauchy sequence $(V(\varepsilon_n, A_n; f_n))\in\mathfrak{C}(f;\nu)$ satisfies the condition (Q*). Hence there exists $F(x)=(E.R.\nu)\int_{-\infty}^{x} f(t)dt$ and it has the approximate derivative $F'_{ap}(x)=f(x)$ for almost all points x of the whole interval.

On the other hand, every interval $I\subset[0, 1]$ contains, for sufficient large n, a interval I_n^i contiguous to H_n and we have

$$\int_{I_n^i}|f(t)|\,dt=\sum_{k=n+1}^{\infty}\frac{2^k}{k}\,\mathrm{mes}\,(I_n^i\cap H_k)=\sum_{k=n+1}^{\infty}\frac{2^k}{k}\frac{\mathrm{mes}\,(I_n^i)}{2^{k-n}}$$

$$=2^n\,\mathrm{mes}\,(I_n^i)\sum_{k=n+1}^{\infty}\frac{1}{k}.$$

Hence $f(x)$ is not summable on any interval and therefore $f(x)$ is not $\mathfrak{D}$-integrable on $[0, 1]$.

References

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