75. On Topological-Additive-Group-Valued Measures

By Masahiro TAKAHASHI

Department of Mathematics, Nara Medical College

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1. Introduction and Summary. A measure has been conceived as a real-valued, usually non-negative set function, whose notable example is the Lebesgue measure. The purpose of this paper is to generalize the notion of a measure to a set function taking values in a topological additive group and to state how such a measure can be extended and completed along the line of the Lebesgue measure theory.

The existence of the Lebesgue measure is, as is well known, verified by at first considering a measure defined for rather simple sets (e.g. half open intervals in one-dimensional case) and next extending it to a measure defined for more complicated sets (i.e. Borel sets or so-called Lebesgue measurable sets), where the outer measure plays an essential role. The construction of the Lebesgue measure is accomplished by completion of the Borel measure.

In this paper, given a ring $\mathcal{R}$, a topological additive group G and a G-valued measure μ on $\mathcal{R}$, we shall define an outer measure, in a generalized sense, μ^* on the *hereditary* (i.e. the conditions $X\in\mathcal{H}$ and $Y\subset X$ imply $Y\in\mathcal{H}$) ring $\mathcal{H}$ generated by $\mathcal{R}$. By using μ^*, the measure μ will be extended to a measure ν on, roughly speaking, a σ-ring $\mathcal{S}$ generated by $\mathcal{R}$. Finally it will be shown that the measure ν on $\mathcal{S}$ can be extended to a measure $\bar{\nu}$ on a σ-ring $\bar{\mathcal{S}}$, which is the completion of ν.

One of the main differences of our theory from the ordinary Lebesgue measure theory is that the 'non-negativity' of real numbers is not available. Difficulties arising from this fact are avoided by replacing the condition with the 'bounded variation' property. Anothor difference is the fact that the group G in which the measure takes values does not admit the element '∞'. We are now dealing with G-valued measures in the strict sense of the term. For this reason, a measure μ on a ring $\mathcal{R}$ can not in general be extended to a measure on, in the strict sense, the σ-ring $\mathcal{S}$ generated by $\mathcal{R}$. So our consideration will be restricted, in place of $\mathcal{S}$, to the class of sets in $\mathcal{S}$ each of which is contained in some set in $\mathcal{R}$.

We shall state the main results with outlines of their proofs throughout the following sections.

2. Extension of a measure. A non-empty class of subsets of a fixed set is called a *ring* if it contains $X \cup Y$ and $X-Y$ provided it contains X and Y. A ring $\mathcal{R}$ will be called a *pseudo-σ-ring* if it contains $\bigcap_{i=1}^{\infty} X_i$, for $X_i \in \mathcal{R}$, $i=1, 2, \cdots$.[1] We shall define a *measure* as a map μ of a ring $\mathcal{R}$ into a topological additive group G satisfying the following conditions:

1) $\mu(X \cup Y)=\mu(X)+\mu(Y)$ for X, Y in $\mathcal{R}$ such that $X \cap Y=\phi$,

2) $\mu(X_i) \to 0$ as $i \to \infty$ for $X_i \in \mathcal{R}$, $i=1, 2, \cdots$, such that $X_i \downarrow \phi$ as $i \to \infty$.[2]

If μ is a G-valued measure on a ring $\mathcal{R}$, and if, for any set X in $\mathcal{R}$ and for any neighbourhood U of the unit element of G, there exists a positive integer n such that if $X \supset X_i \in \mathcal{R}$, $i=1, 2, \cdots, n$, and if $X_j \cap X_k=\phi$ $(j \neq k)$, then there exists an integer i_0 between 1 and n such that $\mu(X_{i_0}) \in U$, then we shall say the measure μ is *of bounded variation.*[3]

Let M be a fixed set and $\mathcal{R}$ be a ring of subsets of M. Then it is easily verified that there exists the smallest pseudo-σ-ring of subsets of M containing $\mathcal{R}$, i.e. the pseudo-σ-ring *generated* by $\mathcal{R}$, which will be denoted by $\mathcal{S}$.

Let us assume G is a Hausdorff, complete topological additive group and μ is a G-valued measure on the above defined ring $\mathcal{R}$.

One of the main purposes of this paper is to establish the following result.

Theorem 1. *If the measure μ is of bounded variation, then μ is uniquely extended to a G-valued measure ν on $\mathcal{S}$ and the extended measure ν is again of bounded variation.*

We shall sketch the outline of the proof of this theorem.

Considering, in the beginning, the class $\tilde{\mathcal{R}}$ of all sets of the form $\bigcup_{i=1}^{\infty} X_i$ which is contained in X_0, where X_i, $i=0, 1, 2, \cdots$, are sets in $\mathcal{R}$, we obtain the following lemma.

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1) This condition is equivalent to the one: $\mathcal{R}$ contains $\bigcup_{i=1}^{\infty} X_i$ if $X_i, i=1, 2, \cdots$, are sets in $\mathcal{R}$ and if $\bigcup_{i=1}^{\infty} X_i \subset X$ for some set X in $\mathcal{R}$.

2) This condition is, under the condition 1), equivalent to each of the following two conditions:

2′) $\mu(X_i) \to \mu(X)$ as $i \to \infty$ for X, X_i in $\mathcal{R}, i=1, 2, \cdots$, such that $X_i \uparrow X$ as $i \to \infty$,

2″) $\mu\left(\bigcup_{i=1}^{\infty} X_i\right)=\sum_{i=1}^{\infty} \mu(X_i)$ for $X_i \in \mathcal{R}, i=1, 2, \cdots$, such that $X_j \cap X_k=\phi (j \neq k)$ and $\bigcap_{i=1}^{\infty} X_i \in \mathcal{R}$,

where '$X_i \downarrow \phi$ as $i \to \infty$' and '$X_i \uparrow X$ as $i \to \infty$' imply that '$X_1 \supset X_2 \supset \cdots$ and $\bigcap_{i=1}^{\infty} X_i=\phi$' and that '$X_1 \subset X_2 \subset \cdots$ and $\bigcup_{i=1}^{\infty} X_i=X$' respectively.

3) A non-negative real-valued measure is always of bounded variation.

Lemma 1. *There exists a unique map $\tilde{\mu}$ of $\tilde{\mathcal{R}}$ into G such that $\mu\left(\bigcup_{i=1}^{n} X_i\right) \to \tilde{\mu}\left(\bigcup_{i=1}^{\infty} X_i\right)$ as $n \to \infty$, if X_i, $i=1, 2, \cdots$, are sets in $\mathcal{R}$ and if $\bigcup_{i=1}^{\infty} X_i \in \tilde{\mathcal{R}}$.*

Proof. Under the assumption that the measure μ is of bounded variation,[4] it can be shown that $\mu\left(\bigcup_{i=1}^{n} X_i\right)$, $n=1, 2, \cdots$, forms a Cauchy sequence in G. Hence, if $X=\bigcup_{i=1}^{\infty} X_i \in \tilde{\mathcal{R}}$, $X_i \in \mathcal{R}$, $i=1, 2, \cdots$, we can, G being complete, define $\tilde{\mu}(X)$ as the limiting point of the sequence $\mu\left(\bigcup_{i=1}^{n} X_i\right)$, $n=1, 2, \cdots$, which is shown to be independent of the choice of X_i's such that $\bigcup_{i=1}^{\infty} X_i = X$. The uniqueness of the map $\tilde{\mu}$ is obvious.

Let us consider the map $\tilde{\mu}$ in Lemma 1 and let $\mathcal{H}$ be the class $\{X \mid X \subset Y$ for some set Y in $\mathcal{R}\}$ of subsets of M (i.e. the hereditary ring generated by $\mathcal{R}$). Then we can prove the following lemma.

Lemma 2. *For any fixed set X in $\mathcal{H}$, $\Gamma(X)=\{Y \mid Y \in \tilde{\mathcal{R}}, Y \supset X\}$ is a directed set, when we write $Y \leqq Z$ if and only if $Y \supset Z$, for Y, Z in $\Gamma(X)$. Moreover, $\tilde{\mu}(Y)$, $Y \in \Gamma(X)$, becomes a Cauchy sequence in G.*

Assigning the limiting point of the sequence $\tilde{\mu}(Y)$, $Y \in \Gamma(X)$, to each X in $\mathcal{H}$, we have the following lemma.

Lemma 3. *There exists a unique map μ^* of $\mathcal{H}$ into G having the following property: if $X \in \mathcal{H}$, $Y \in \tilde{\mathcal{R}}$ and $X \subset Y$, then, for any neighbourhood U of the unit element of G, we can find a set Z in $\tilde{\mathcal{R}}$ such that $X \subset Z \subset Y$ and $\mu(Z)-\mu^*(X) \in U$.*

Thus we can define a map μ^* of the hereditary ring $\mathcal{H}$ generated by $\mathcal{R}$ into G, which is anologous to (but not the same with) the *outer measure* in the Lebesgue measure theory.

Let $\mathcal{S}'$ be the subclass of $\mathcal{H}$ defined by $\{X \mid X \in \mathcal{H}, \mu^*(Y)= \mu^*(Y \cap X)+\mu^*(Y-X)$ for any set Y in $\mathcal{H}\}$, which may be understood to be the class of the *measurable* sets with respect to μ^*.

Then it can be verified that $\mathcal{S}'$ is a pseudo-σ-ring containing (but, in general, not generated by) $\mathcal{R}$ and that the map μ^* has the following properties:

Lemma 4. *$\mu^*(X)=\mu(X)$ for any X in $\mathcal{R}$.*

Lemma 5. *$\mu^*(X \cup Y)=\mu^*(X)+\mu^*(Y)$ for X, Y in $\mathcal{S}'$ such that $X \cap Y=\phi$.*

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4) For our present purpose to prove Lemma 1, this condition can be replaced by a slightly weaker one: if $X \in \mathcal{R}$, $X_i \in \mathcal{R}$, $X \supset X_i$, $i=1, 2, \cdots$, and $X_j \cap X_k=\phi$ for $j \neq k$, then, for any neighbourhood U of the unit element of G, we can find a positive integer i_0 such that $\mu(X_i) \in U$ for any $i \geqq i_0$. The bounded variation property of the measure μ is used in proving Lemma 2.

Lemma 6. *$\mu^*(X_i) \to \mu^*(X)$ as $i \to \infty$ for X, X_i in $\mathcal{H}$, $i=1, 2, \cdots$, such that $X_i \uparrow X$ as $i \to \infty$.*[5),6)]

Corollary. *$\mu^*(X_i) \to 0$ as $i \to \infty$ for $X_i \in \mathcal{S}'$, $i=1, 2, \cdots$, such that $X_i \downarrow \phi$ as $i \to \infty$.*

This implies that the restriction ν' of μ^* to $\mathcal{S}'$ is a G-valued measure defined on the pseudo-σ-ring $\mathcal{S}'$. Since $\mathcal{S}$ is contained in $\mathcal{S}'$, the restriction ν of μ^* on $\mathcal{S}$ is also a measure. Lemma 4 shows that these measures are extensions of μ.

Thus we can prove that there exists at least one G-valued measure ν on $\mathcal{S}$ which is an extension of μ. It is also verified that such an extension is uniquely determined and that the measure ν is of bounded variation, which accomplishes the verification of the theorem.

Remark. As is seen above, there exists a measure ν' on the pseudo-σ-ring $\mathcal{S}'$ which is an extension of μ (and is also an extension of ν), but the uniqueness of ν' no longer holds for $\mathcal{S}'$.

3. Completion of a measure. Let the notations in section 2 be reserved and let $\mathfrak{N}$ be the class of all sets N such that $N \subset X$ for some set X belonging to the class $\mathfrak{N}_0 = \{X \mid X \in \mathcal{S}, \nu(Y)=0$ for any set Y in $\mathcal{S}$ such that $Y \subset X\}$. Then we have the following theorem.

Theorem 2. *Let $\bar{\mathcal{S}}$ be the class $\{(X-N) \cup (N-X) \mid X \in \mathcal{S}, N \in \mathfrak{N}\}$. Then $\bar{\mathcal{S}}$ is a pseudo-σ-ring containing $\mathcal{S}$ together with $\mathfrak{N}$ and there exists a unique G-valued measure $\bar{\nu}$ on $\bar{\mathcal{S}}$ satisfying the following conditions:*

1) $\bar{\nu}(X) = \nu(X)$ *if* $X \in \mathcal{S}$,

2) $\bar{\nu}(N) = 0$ *if* $N \in \mathfrak{N}$.

Before giving the proof, we shall give some remarks which are well known. Defining $X+Y$ and XY by $(X-Y) \cup (Y-X)$ and $X \cap Y$, respectively, for each pair X, Y of subsets of M, the class $\mathcal{M}$ of all the subsets of M becomes a ring in the algebraic sense of the word, and a class $\mathcal{A}$ of subsets of M is a ring in the set theoretical sense if and only if $\mathcal{A}$ is an algebraic subring of $\mathcal{M}$.

In the terminology in the above remarks, it can be seen that $\mathfrak{N}$ is an ideal of the ring $\mathcal{M}$, so that $\bar{\mathcal{S}}$, which may be written as a sum of a subring $\mathcal{S}$ and an ideal $\mathfrak{N}$ of $\mathcal{M}$, is a subring of $\mathcal{M}$ and consequently is a ring in the set theoretical sense.

Proof of the theorem. The proof of the fact that $\bar{\mathcal{S}}$ is a ring being given above, we shall show that $\bar{\mathcal{S}}$ contains $\bigcap_{i=1}^{\infty} X_i$ for $X_i \in \bar{\mathcal{S}}$, $i=1, 2, \cdots$, which assures us that $\bar{\mathcal{S}}$ is a pseudo-σ-ring.

5) Refer to the footnote 2).

6) A *regular* outer measure in the Lebesgue measure theory has this property ([1] p. 53).

By definition, we can write, for $i=1, 2, \cdots, X_i = Y_i + N_i$, where Y_i is a set in $\mathcal{S}$ and N_i is a subset of M which is contained in some set Z_i in $\mathcal{S}$ of which any subset W_i belonging to $\mathcal{S}$ satisfies the condition $\nu(W_i)=0$. The formula $\bigcap_{i=1}^{\infty} X_i \subset Y_1 \cup Z_1 \in \mathcal{S}$ implies that we may assume that N_i, $i=1, 2, \cdots$, are contained in some fixed set in $\mathcal{S}$. Then we have $N \in \mathfrak{N}$ if we write $N = \bigcup_{i=1}^{\infty} N_i$. When $\bigcap_{i=1}^{\infty} X_i$ and $\bigcap_{i=1}^{\infty} Y_i$ are denoted by X and Y respectively, we get $X = \bigcap_{i=1}^{\infty} ((Y_i - N_i) \cup (N_i - Y_i)) \subset \bigcap_{i=1}^{\infty} (Y_i \cup N) = Y \cup N$ and $X \supset \bigcap_{i=1}^{\infty} (Y_i - N) = Y - N$, which give us $X - Y \subset N$ and $Y - X \subset N$. Thus we have $X + Y = (X - Y) \cup (Y - X) \subset N$ which implies that $X + Y \in \mathfrak{N}$, and accordingly we have $\bigcap_{i=1}^{\infty} X_i = X = Y + (X + Y) \in \mathcal{S} + \mathfrak{N} = \bar{\mathcal{S}}$. The fact that $\bar{\mathcal{S}}$ contains $\mathcal{S}$ and $\mathfrak{N}$ is obvious.

The existence of the measure $\bar{\nu}$ is shown as follows. It is easily seen that $\mathfrak{N}_0 = \mathcal{S} \cap \mathfrak{N}$ and that there exists a unique map ν_0 of the residue class ring $\mathcal{S}/\mathfrak{N}_0$ into G such that $\nu_0(\bar{X}) = \nu(X)$ if the residue class $\bar{X}$ contains X. Let φ be the canonical isomorphism of $\bar{\mathcal{S}}/\mathfrak{N} = (\mathcal{S} + \mathfrak{N})/\mathfrak{N}$ onto $\mathcal{S}/\mathfrak{N}_0 = \mathcal{S}/(\mathcal{S} \cap \mathfrak{N})$. Putting, for $X \in \bar{\mathcal{S}}$, $\bar{\nu}(X) = \nu_0(\varphi(\bar{X}))$, where $\bar{X}$ is the residue class containing X, we obtain the measure $\bar{\nu}$ required. The uniqueness of the measure $\bar{\nu}$ is clear and thus the theorem is proved.

This measure $\bar{\nu}$, which corresponds to the *completion* of ν in the Lebesgue measure theory, can be proved to be of bounded variation. It is also seen that $\mathfrak{N}$ coincides with the class $\{X \mid X \in \bar{\mathcal{S}}, \bar{\nu}(Y)=0$ for any set Y in $\bar{\mathcal{S}}$ such that $Y \subset X\}$, which implies that the completion $\bar{\bar{\nu}}$ of $\bar{\nu}$ concides with $\bar{\nu}$.[7]

We shall state the following theorem without proof.

Theorem 3. *If G satisfies the first condition of countability, then $\bar{\mathcal{S}}$ and $\bar{\nu}$ coincide with $\mathcal{S}'$ and μ^* (strictly speaking, the restriction of μ^* on $\mathcal{S}'$) respectively.*

We shall close this paper by noticing that the Lebesgue measure can be constructed along the line of these theorems taking some sets 'of measure ∞' into consideration.

Reference

[1] P. R. Halmos: Measure Theory. Van Nostrand (1950).

7) The *completeness* of a measure may be defined as follows: given a measure λ on a ring $\mathcal{P}$ taking values in a topological additive group F, the measure λ is complete if the ring $\mathcal{P}$ contains the class $\mathcal{L} = \{X \mid X \in \mathcal{P}, \lambda(Y)=0$ for any set Y in $\mathcal{P}$ such that $Y \subset X\}$. Then our measure $\bar{\nu}$ is a complete measure which is an extention of ν.