135. The Cesàro-Perron-Stieltjes Integral. I

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1. Introduction. The Cesàro-Perron integral was introduced by J. C. Burkill [1] by means of major and minor functions using inequalities relating to Cesàro-derivates. In extending such a definition to the Stieltjes type of integration with respect to a general function which may attain the same value at an infinite set, there would be difficulties. We shall define the Cesàro-Perron-Stieltjes integral (*CPS*-integral) by the method of A. J. Ward [5] which uses inequalities concering the increments directly and not in terms of derivates with respect to a function.

The resulting integral is essentially an extension of the Cesàro-Perron integral and we shall prove some continuous and differential properties of the indefinite *CPS*-integral. However the relationship between our integral and the *PS*-integral of A. J. Ward is still open.

2. Cesàro-continuity and Cesàro-derivates with respect to a function. Let $f(x)$, $\varphi(x)$ be real valued (finite) functions defined on the interval $[a, b]$. We say that $f(x)$ is *Cesàro-continuous with respect to* $\varphi(x)$ at the point x_0, if for some number K

$$(1)\qquad \lim_{x\to x_0}\left\{C(f, x_0, x)-f(x_0)-\frac{1}{2}K[\varphi(x)-\varphi(x_0)]\right\}=0,$$

where we put

$$C(f, a, b)=\frac{1}{b-a}\int_a^b f(t)\,dt,$$

the integral being taken in the special Denjoy sense. If in addition we have

$$(2)\qquad \overline{\lim_{x\to x_0+0}}\left\{C(f, x_0, x)-f(x_0)-\frac{1}{2}K[\varphi(x)-\varphi(x_0)]\right\}\Big/\omega(\varphi, [x_0, x])=0$$

then we say that the *right-hand Cesàro-Roussel derivate of* $f(x)$ *with respect to* $\varphi(x)$ at x_0 is K, where $\omega(\varphi, [x_0, x])$ denotes the oscillation of $\varphi(x)$ on $[x_0, x]$, and write $\overline{CD}_+(f, x_0, \varphi)=K$. The ratio in (2) is to be interpreted to mean 0 whenever its numerator and denominator vanish together. When the oscillation of φ is finite, the condition (2) evidently implies (1); however when $\omega(\varphi, [x_0, x])=+\infty$, the condition (1) plays an essential part.

We define three other derivates similarly and put

$$\overline{CD}(f, x_0, \varphi) = \max\{\overline{CD}_+(f, x_0, \varphi), \overline{CD}_-(f, x_0, \varphi)\},$$
$$\underline{CD}(f, x_0, \varphi) = \min\{\underline{CD}_+(f, x_0, \varphi), \underline{CD}_-(f, x_0, \varphi)\}.$$

If they are equal then we write the common value as $CD(f, x_0, \varphi)$.

3. The Cesàro-Perron-Stieltjes integral. First we define the major and minor functions of $f(x)$ with respect to $\varphi(x)$ on $[a, b]$.

Definition 1. We say that $M(x)$ is a *major* function of $f(x)$ with respect to $\varphi(x)$ if (i) $M(a)=0$,
and

(ii) for any point x of $[a, b]$ there exists a number $\delta(x)>0$ such that

(1) $$C(M, x, t)-M(x) \geqq \frac{1}{2} f(x)[\varphi(t)-\varphi(x)] \qquad \text{if } 0<t-x\leqq\delta(x),$$

(2) $$C(M, x, t)-M(x) \leqq \frac{1}{2} f(x)[\varphi(t)-\varphi(x)] \qquad \text{if } -\delta(x)\leqq t-x<0.$$

Definition 2. We say that $m(x)$ is a *minor* function of $f(x)$ with respect to $\varphi(x)$ if (i) $M(a)=0$,
and

(ii) for any point x of $[a, b]$ there exists a number a $\delta(x)>0$ such that

(3) $$C(m, x, t)-m(x) \leqq \frac{1}{2} f(x)[\varphi(t)-\varphi(x)] \qquad \text{if } 0<t-x\leqq\delta(x),$$

(4) $$C(m, x, t)-m(x) \geqq \frac{1}{2} f(x)[\varphi(t)-\varphi(x)] \qquad \text{if } -\delta(x)\leqq t-x<0.$$

Now we state an important Lemma to develope the theory which is due to G. Sunouchi and M. Utagawa [4].

Lemma 1. *If $f(x)$ is a measurable function defined on $[a, b]$ and $\underline{CD}f(x)\geqq 0$ at each point x of $[a, b]$ then $f(x)$ is non-decreasing on $[a, b]$, where $\underline{CD}f(x)$ denotes the ordinary lower Cesàro-derivate of $f(x)$ at x.*

Theorem 1. *For any major and minor functions $M(x)$, $m(x)$ of $f(x)$ with respect to $\varphi(x)$ on $[a, b]$, the function $M(x)-m(x)$ is non-decreasing on $[a, b]$.*

Proof. We put $\omega(x)=M(x)-m(x)$. Then it follows from (1) and (3) that for any point x of $[a, b]$, there exists $\delta(x)>0$ such that

$$C(M, x, t)-M(x) \geqq \frac{1}{2} f(x)[\varphi(t)-\varphi(x)],$$

and

$$C(m, x, t)-m(x) \leqq \frac{1}{2} f(x)[\varphi(t)-\varphi(x)], \qquad \text{for } 0<t-x\leqq\delta(x).$$

Therefore
$$C(M, x, t)-M(x) \geqq C(m, x, t)-m(x) \qquad \text{for } 0<t-x\leqq\delta(x).$$

That is,

$$\frac{1}{t-x}\int_x^t [M(t)-M(x)]dt \geqq \frac{1}{t-x}\int_x^t [m(t)-m(x)]dt,$$

for $0<t-x\leqq\delta(x)$.
Consequently we have

$$\frac{1}{t-x}\int_x^t [\omega(t)-\omega(x)]dt \geqq 0 \qquad \text{for } x<t\leqq x+\delta(x).$$

That is, for $x<t\leqq x+\delta(x)$

$$\frac{1}{t-x}\int_x^t [\omega(t)-\omega(x)]dt \Big/ \frac{1}{2}(t-x) \geqq 0.$$

Hence we obtain

$$\underline{CD}_+\omega(x)\geqq 0.$$

Similarly we have from (2) and (4)

$$\underline{CD}_-\omega(x)\geqq 0,$$

and therefore

$$\underline{CD}\omega(x)\geqq 0.$$

It follows from Lemma 1 that $\omega(x)$ is non-decreasing on $[a, b]$.

Definition 3. If a function $f(x)$ has major and minor functions $M(x)$, $m(x)$ with respect to $\varphi(x)$ on $[a, b]$ and if

$$\inf_M M(b) = \sup_m m(b)$$

then $f(x)$ is termed *integrable in the Cesàro-Perron-Stieltjes sense with respect to* $\varphi(x)$ or *CPS*-integrable with respect to $\varphi(x)$ and we denote the common value by $(CPS)\int_a^b f(t)d\varphi(t)$ or $(CPS)\int_a^b fd\varphi$.

We can now prove the following theorems as usual.

Theorem 2. *If $f(x)$ is CPS-integrable with respect to $\varphi(x)$ on $[a, b]$ then $f(x)$ is also so in every sub-interval $[a, x]$ for $a<x\leqq b$.*

Theorem 3. *For indefinite integral $F(x)$,*

$$F(x)=(CPS)\int_a^x f(t)d\varphi(t)$$

and any major and minor functions $M(x), m(x)$, the functions $M(x)-F(x)$, and $F(x)-m(x)$ are both non-decreasing on $[a, b]$.

Theorem 4. (i) *If $f(x)$ is CPS-integrable with respect to $\varphi(x)$ on $[a, b]$ then for $a<c<b$,*

$$(CPS)\int_a^b fd\varphi=(CPS)\int_a^c fd\varphi+(CPS)\int_c^b fd\varphi.$$

(ii) *If $f(x)$ and $g(x)$ are CPS-integrable with respect to $\varphi(x)$ on $[a, b]$ then $\alpha f+\beta g$ is also so and*

$$(CPS)\int_a^b (\alpha f+\beta g)d\varphi=\alpha(CPS)\int_a^b fd\varphi+\beta(CPS)\int_a^b gd\varphi.$$

4. The properties of the indefinite *CPS*-integral.

Theorem 5. *The indefinite CPS-integral of $f(x)$ with respect* *to $\varphi(x)$ is Cesàro-continuous with respect to $\varphi(x)$ on $[a, b]$.*

Proof. Given $\varepsilon>0$, we can choose the major function $M(x)$ such that

$$M(b)<F(b)+\varepsilon.$$

Since $M(x)-F(x)$ is non-decreasing on $[a, b]$, we have

$$F(t)-F(x)\geqq M(t)-M(x)-\varepsilon \qquad \text{for } t>x.$$

Consequently we obtain

$$\frac{1}{t-x}\int_x^t [F(t)-F(x)]dt \geqq \frac{1}{t-x}\int_x^t [M(t)-M(x)]dt-\varepsilon \qquad \text{for } t>x,$$

that is,

$$C(F, x, t)-F(x)\geqq C(M, x, t)-M(x)-\varepsilon \qquad \text{for } t>x.$$

Hence, for $0<t-x\leqq\delta(x)$, we have from (1)

$$C(F, x, t)-F(x)\geqq\frac{1}{2}f(x)[\varphi(t)-\varphi(x)]-\varepsilon.$$

Similarly we obtain using minor functions

$$C(F, x, t)-F(x)\leqq\frac{1}{2}f(x)[\varphi(t)-\varphi(x)]+\varepsilon$$

for $-\delta(x)\leqq t-x<0$.
Therefore

$$\left|C(F, x, t)-F(x)-\frac{1}{2}f(x)[\varphi(t)-\varphi(x)]\right|<\varepsilon$$

for $0<|t-x|\leqq\delta(x)$ which completes the proof.

Lemma 2 (A. J. Ward [5]). *Let E be any linear set. If with each point x of E an interval $(x, x+h)$, h depending on x, is associated then given any number A $(A<m_e\varphi(E))$, we can find a finite non-overlapping set of intervals (x_k, x_k+h_k) such that*

$$\sum m_e\varphi[E(x_k, x_k+h_k)]>A.$$

Theorem 6. *If*

$$F(x)=(CPS)\int_a^x f(t)d\varphi(t) \qquad (a\leqq x\leqq b)$$

then $CD(F, x, \varphi)=f(x)$ except at points of a set E such that $m\varphi(E)=0$.

Proof. If $\varphi(x)$ is constant on $[a, b]$, then $F(x)$ is also constant, so that the equation $CD(F, x, \varphi)=f(x)$ is true in a conventional sense.

Now we consider the set E_1 of points x_0 such that $\varphi(x)$ is not constant in any interval $[x_0, x]$ and that

$$(1) \quad \overline{\lim_{x\to x_0+0}} \left\{C(F, x_0, x)-F(x_0)-\frac{1}{2}f(x_0)[\varphi(x)-\varphi(x_0)]\right\}\Big/\omega(\varphi, [x_0, x])>0.$$

We shall show that $m\varphi(E_1)=0$. Suppose that $m_e\varphi(E_1)>0$. Then we can find a natural number p such that the E_p consisting of points

x_0 at which

$$\overline{\lim_{x\to x_0+0}}\left\{C(F, x_0, x)-F(x_0)-\frac{1}{2}f(x_0)[\varphi(x)-\varphi(x_0)]\right\}\Big/\omega(\varphi, [x_0, x])>1/p,$$

satisfies $m_e\varphi(E_p)>0$. Take ε such that $0<\varepsilon<m_e\varphi(E_p)$ and a minor function $m(x)$ with

$$(2)\qquad F(b)-m(b)<\varepsilon/p.$$

Since

$$C(m, x_0, x)-m(x_0)\leqq\frac{1}{2}f(x_0)[\varphi(x)-\varphi(x_0)]$$

for all x sufficiently near to x_0 $(x>x_0)$, we have, for x_0 in E_p,

$$\overline{\lim_{x\to x_0+0}}\left\{C(F, x_0, x)-C(m, x_0, x)-[F(x_0)-m(x_0)]\right\}\Big/\omega(\varphi, [x_0, x])>1/p,$$

and therefore, for $x_0\in E_p$ and sufficiently small $h_0>0$,

$$C(F, x_0, x_0+h_0)-C(m, x_0, x_0+h_0)$$
$$-[F(x_0)-m(x_0)]>1/p\cdot\omega(\varphi, [x_0, x_0+h_0]).$$

Applying Lemma 2 to the set E_p, we can find a finite non-overlapping set of intervals (x_k, x_k+h_k) $(k=1, 2, \cdots, n)$ such that

$$(3)\qquad C(F, x_k, x_k+h_k)-C(m, x_k, x_k+h_k)$$
$$-F(x_k)-m(x_k)>1/p\cdot\omega(\varphi, [x_k, x_k+h_k])$$

and

$$\sum_{k=1}^{n} m_e\varphi[E_p(x_k, x_k+h_k)]>\varepsilon.$$

Since

$$\sum_{k=1}^{n}\omega(\varphi, [x_k, x_k+h_k])\geqq\sum_{k=1}^{n} m_e\varphi[E_p(x_k, x_k+h_k)],$$

we have from (3)

$$(4)\qquad \sum_{k=1}^{n}\frac{1}{h_k}\int_{x_k}^{x_k+h_k}[F(t)-m(t)]dt-\sum_{k=1}^{n}[F(x_k)-m(x_k)]>\frac{\varepsilon}{p}.$$

The function $F(x)-m(x)$ is non-decreasing (by Theorem 3) and (x_k, x_k+h_k) is non-overlapping, so that we obtain from (4)

$$F(b)-m(b)>\varepsilon/p,$$

which is in contradiction to (2). Thus $m\varphi(E_1)=0$.

Similar argument applied to three other sets defined by inequalities analogous to (1) would complete the proof of the theorem, for we have already shown (Theorem 5) that $F(x)$ is Cesàro-continuous with respect to $\varphi(x)$ at every point.

Next we shall prove that the *CPS*-integral is essentially an extension of the ordinary Cesàro-Perron integral (*CP*-integral).

For any not necessarily finite function $f(x)$ on $[a, b]$, we define the function $\bar{f}(x)$ which is equal to $f(x)$ if $f(x)$ is finite and equal to 0 elsewhere.

Theorem 6. *If $f(x)$ is CP-integrable on $[a, b]$, then $\bar{f}(x)$ is*

CPS-integrable with respect to $\varphi(x)=x$ *and*

$$(1) \qquad (CP)\int_a^b f(t)dt=(CPS)\int_a^b \bar{f}(t)d\varphi(t).$$

Proof. Since $f(x)$ is CP-integrable on $[a, b]$, $f(x)$ is finite almost everywhere. Hence $\bar{f}(x)$ is also CP-integrable on $[a, b]$ and

$$(2) \qquad (CP)\int_a^b \bar{f}(t)dt=(CP)\int_a^b f(t)dt.$$

Given any $\varepsilon>0$ we can find a ordinary major function $M(x)$ for $\bar{f}(x)$ with $M(a)=0$ such that

$$\underline{CD}M(x)\geqq \bar{f}(x)$$

everywhere and such that

$$M(b)<(CP)\int_a^b \bar{f}(t)dt+\varepsilon.$$

We consider the function

$$M_1(x)=M(x)+\varepsilon(x-a)/(b-a).$$

Since $\underline{CD}M_1(x)>\bar{f}(x)$, we have for sufficiently small $\delta(x)>0$

$$\left\{C(M_1, x, t)-M_1(x)\right\}\Big/\frac{1}{2}(t-x)>\bar{f}(x) \qquad \text{if } 0<|x-t|\leqq\delta(x).$$

Thus $M_1(x)$ is a major function of $\bar{f}(x)$ with respect to $\varphi(x)=x$, and

$$M_1(b)<(CP)\int_a^b \bar{f}(t)dt+2\varepsilon.$$

Similarly we can find a minor function $m_1(x)$ with respect to $\varphi(x)=x$ such that

$$(CP)\int_a^b \bar{f}(t)dt-2\varepsilon<m_1(b).$$

Hence

$$0\leqq M_1(b)-m_1(b)<4\varepsilon.$$

Therefore $\bar{f}(x)$ is CPS-integrable with respect to $\varphi(x)=x$ and

$$(CP)\int_a^b \bar{f}(t)dt=(CPS)\int_a^b f(t)d\varphi(t),$$

where $\varphi(x)=x$, which together with (2) implies (1).

References

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