123. Notes on Commutative Archimedean Semigroups. II

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This note is the continuation of [1] to report the results without proof. The same notations as those in [1] will be used without explanation.

6. Construction of the semigroups without idempotent.

Definition. An ordinary tree is a dispersed tree which satisfies the ascending chain condition and has at least one highest prime. An ordinary tree without smallest element is also called an ordinary tree of infinite length.

Theorem 8. *Assume that the following systems and functions are given:*

(15.1) *An abelian group G with a function I satisfying* (1.1) *through* (1.4).

(15.2) *A family $\{S_\lambda; \lambda \in G\}$ of ordinary trees of infinite length.*

(15.3) *A set $\{\iota_\lambda; \lambda \in G\}$ of highest primes.*

(15.4) *A commutative groupoid $(\cdot)$, $P=\bigcup_{\lambda\in G} P_\lambda$ with identity ι_ε where P_λ is the set of all primes of S_λ such that*

$$\text{for } \alpha_\lambda \in P_\lambda \text{ and } \beta_\mu \in P_\mu,\ \alpha_\lambda\cdot\beta_\mu \in P_{\lambda\mu}$$

and the following conditions are satisfied:

(16.1) $\sigma(\alpha_\lambda)+\sigma(\beta_\mu)+I(\lambda, \mu)-\sigma(\alpha_\lambda\cdot\beta_\mu)$
$\geqq h_{\alpha_\lambda\cdot\beta_\mu}(\alpha_\lambda\cdot\beta_\mu, \alpha_\lambda\cdot\beta'_\mu)-h_{\beta_\mu}(\beta_\mu, \beta'_\mu)$
for all $\alpha_\lambda \in P_\lambda$, $\beta_\mu, \beta'_\mu \in P_\mu$, *all* $\lambda, \mu \in G$.

(16.2) $\sigma(\alpha_\lambda)+\sigma(\beta_\mu)+\sigma(\gamma_\nu)+I(\lambda, \mu)+I(\lambda\mu, \nu)$
$\geqq \sigma((\alpha_\lambda\cdot\beta_\mu)\cdot\gamma_\nu)+h_{(\alpha_\lambda\cdot\beta_\mu)\cdot\gamma_\nu}((\alpha_\lambda\cdot\beta_\mu)\cdot\gamma_\nu, \alpha_\lambda\cdot(\beta_\mu\cdot\gamma_\nu))$
for all $\alpha_\lambda \in P_\lambda$, $\beta_\mu \in P_\mu$, $\gamma_\nu \in P_\nu$, $\lambda, \mu, \nu \in G$.

(16.3) *For any $\alpha_\lambda \in P_\lambda$ there is $m>0$ such that*
$\sigma(\alpha_\lambda^{(m)})+\sigma(\alpha_\lambda)+I(\lambda^m, \lambda)-\sigma(\alpha_\lambda^m\cdot\alpha_\lambda)>0$
where $\alpha_\lambda^{(m)}=\alpha_\lambda^{(m-1)}\cdot\alpha_\lambda$, $\alpha_\lambda^{(2)}=\alpha_\lambda\cdot\alpha_\lambda$

(16.1) *implies* (16.4) *below*:

(16.4) $\sigma(\alpha_\lambda)+\sigma(\beta_\mu)+I(\lambda, \mu)\geqq\sigma(\alpha_\lambda\cdot\beta_\mu)$.

Now we define a function $K(\alpha_\lambda, \beta_\mu)$ on $P\times P$ as follows:

(16.5) $K(\alpha_\lambda, \beta_\mu)=\sigma(\alpha_\lambda)+\sigma(\beta_\mu)+I(\lambda, \mu)-\sigma(\alpha_\lambda\cdot\beta_\mu)$.

Let $N\times P=\{(n, \alpha): n\in N, \alpha\in P\}$ and let $S=(N\times P)/\xi$ where ξ is an equivalence defined by

$(n, \alpha)\xi(m, \beta)$ *if and only if α and β are in a same S_λ and*

$$n-h_\alpha(\alpha, \beta)=m-h_\beta(\alpha, \beta)\geqq 0.$$

Let $[(n, \alpha)]$ *denote an element of* S, *i.e., the equivalence class (modulo* ξ*) containing* (n, α). *Finally we define a binary operation on* S *as follows*:

$$[(n, \alpha_\lambda)][(m, \beta_\mu)]=[(n+m+K(\alpha_\lambda, \beta_\mu), \alpha_\lambda\cdot\beta_\mu)].$$

Then S *is a commutative archimedean semigroup without idempotent. Every commutative archimedean semigroup without idempotent can be obtained in this manner.*

Definition. We call system $((G, I; \{S_\lambda\}, \{\iota_\lambda\}; P))$ a structural system if the conditions (16.1), (16.2), (16.3) are satisfied.

Remark.

(17) $\sigma(\alpha_\lambda)+\sigma(\beta_\mu)+I(\lambda, \mu)\geqq\pi(\alpha_\lambda\cdot\beta_\mu)$ for all $\alpha_\lambda\in P_\lambda$, all $\beta_\mu\in P_\mu$, and all $\lambda, \mu\in G$, implies (16.1), (16.2), (16.3).

Proposition. $\sigma(\alpha_\lambda)+\sigma(\beta_\mu)+I(\lambda, \mu)\geqq\pi(\alpha_\lambda\cdot\beta_\mu)$ *identically holds if and only if*

(18) $\alpha_\lambda\neq\iota_\lambda$ *or* $\beta_\mu\neq\iota_\mu$ *implies*

$$\sigma(\alpha_\lambda)+\sigma(\beta_\mu)+I(\lambda, \mu)-\sigma(\alpha_\lambda\cdot\beta_\mu)\geqq 1.$$

(19) $\pi(\alpha_\lambda)=\pi(\alpha_\lambda)-\sigma(\alpha_\lambda)=1$ *for all* $\alpha_\lambda\neq\iota_\lambda$, *all* $\lambda\in G$.

Definition. If an ordinary tree satisfies (19), then it is called a sparse tree. If each S_λ is a sparse tree and if G, I, and S_λ satisfy (17), then $((G, I; \{S_\lambda\}, \{\tau_\lambda\}; P))$ is called a sparse structural system.

We have the following existence theorems of structural system.

Theorem 9. *Suppose that* G *and* I *are given, and that* $\{S_\lambda; \lambda\in G\}$ *is a family of sparse trees (without any restriction between* S_λ *and* S_μ $(\lambda\neq\mu)$*). Let* P_λ *be the set of all primes of* S_λ *and* $\{\iota_\lambda; \lambda\in G\}$ *be a representative system of highest primes. Then there is a commutative groupoid* P *with identity* ι_ε *such that* $((G, I; \{S_\lambda\}, \{\tau_\lambda\}; P))$ *is a sparse structural system.*

Theorem 10. *Suppose that* G *is given and that there is given a family of disjoint sets* $\{P_\lambda; \lambda\in G\}$ *in which* $\{\iota_\lambda; \lambda\in G\}$, $\iota_\lambda\in P_\lambda$, *is assigned. Let* $P=\bigcup_{\lambda\in G} P_\lambda$. *Also assume that a groupoid operation is given such that*

$$\alpha_\lambda\cdot\beta_\mu=\beta_\mu\cdot\alpha_\lambda\in P_{\lambda\mu},\ \alpha_\lambda\in P_\lambda,\ \beta_\mu\in P_\mu$$

$$\iota_\varepsilon\cdot\alpha_\lambda=\alpha_\lambda\cdot\iota_\varepsilon=\alpha_\lambda \text{ for all } \alpha_\lambda\in P_\lambda, \text{ all } \lambda\in G.$$

Then there is a function I *and a family* $\{S_\lambda; \lambda\in G\}$ *of sparse trees such that the elements of* P_λ *are primes of* S_λ *and* ι_λ *is a highest prime in* S_λ, *and* $((G; I; \{S_\lambda\}, \{\iota_\lambda\}; P))$ *is a sparse structural system.*

7. Construction of the semigroups with zero. In this section let S be a commutative archimedean semigroup with zero; hence S satisfies for any element x of S, $x^m=0$ for some $m>0$ depending on

x. We call S also a commutative nilsemigroup. For construction of S all the arguments in the preceding section are not effective since the congruence η is trivial and the function K can not be defined by using I. Therefore, we have to find K directly, but we notice that there is a non-zero element a such that the tree with respect to τ_a is an ordinary tree whose length is at most 2.

Theorem 11. *Assume that there is given an ordinary tree L whose length is at most 2. Let P be the set of all primes and ι be a prime such that $w(\iota)=1$. Also assume that a commutative groupoid P with identity ι is arbitrarily given. Choose a non-negative integer valued function $K(\alpha, \beta)$ such that the following conditions are satisfied; $\alpha\cdot\beta$ denotes the product in the groupoid P.*

I. $0\leq K(\alpha, \beta)\leqq 2$.

II. $K(\alpha, \beta)=K(\beta, \alpha)$, $K(\alpha, \varepsilon)=1$.

III. *If* $w(\alpha)<w(\alpha')=w(\alpha\cdot\beta)$ *or* $w(\alpha\cdot\beta)>w(\alpha'\cdot\beta)$, *then* $K(\alpha, \beta)\geqq 1$.

If $w(\alpha)=w(\alpha')$ *and if* $w(\alpha\cdot\beta)=w(\alpha'\cdot\beta)=2$ *and*
$$h_{\alpha\cdot\beta}(\alpha\cdot\beta, \alpha'\cdot\beta)=h_{\alpha'\cdot\beta}(\alpha\cdot\beta, \alpha'\cdot\beta)=2, \text{ then}$$
$$K(\alpha, \beta)\equiv K(\alpha', \beta)\equiv 1 \quad (mod.\ N_1).^{*)}$$
If $w(\alpha)=w(\alpha')$ *and* $w(\alpha\cdot\beta)=w(\alpha'\cdot\beta)=2$ *and if*
$$h_{\alpha\cdot\beta}(\alpha\cdot\beta, \alpha'\cdot\beta)=h_{\alpha'\cdot\beta}(\alpha\cdot\beta, \alpha'\cdot\beta)=1 \text{ or } 0, \text{ then}$$
$$K(\alpha, \beta)\equiv K(\alpha', \beta) \quad (mod.\ N_1).$$

IV. *If* $w((\alpha\cdot\beta)\cdot\gamma)\neq w(\alpha\cdot(\beta\cdot\gamma))$, *then*
$$K(\alpha, \beta)+K(\alpha\cdot\beta, \gamma)\geqq w((\alpha\cdot\beta)\cdot\gamma) \text{ and}$$
$$K(\alpha, \beta\gamma)+K(\beta, \gamma)\geqq w(\alpha\cdot(\beta\cdot\gamma)).$$
If $w((\alpha\cdot\beta)\cdot\gamma)=w(\alpha\cdot(\beta\cdot\gamma))$, *then*
$$K(\alpha, \beta)+K(\alpha\cdot\beta, \gamma)\equiv K(\alpha, \beta\cdot\gamma)+K(\beta, \gamma) \quad (mod.\ N_l)$$
where $l=w((\alpha\cdot\beta)\cdot\gamma)-h_{(\alpha\cdot\beta)\cdot\gamma}((\alpha\cdot\beta)\cdot\gamma, \alpha\cdot(\beta\cdot\gamma))$.

V. *For any* $\alpha\in P$, *there is* $m>0$ *such that* $K(\alpha^{(m)}, \alpha)>0$.

In particular if $\alpha\cdot\alpha=\alpha$, *then* $K(\alpha, \alpha)>0$. *Such a function* $K(\alpha, \beta)$ *always exists.*

Let $S'=\{(n, \alpha);\ n=0, 1, 2, \cdots;\ \alpha\in P\}$ *and* $S=S'/\xi$ *where* ξ *is defined by*
$$(n, \alpha)\xi(m, \beta) \quad \text{if and only if}$$
either $n\geqq w(\alpha)$ *and* $m\geqq w(\beta)$

or $n<w(\alpha), m<w(\beta)$, *and* $n-h_\alpha(\alpha, \beta)=m-h_\beta(\alpha, \beta)\geqq 0$.

A binary operation is defined on S *as follows:*
$$[(n, \alpha)][(m, \beta)]=[(n+m+K(\alpha, \beta), \alpha\cdot\beta)].$$
Then S is a commutative archimedean semigroup with zero. Any semigroup of this kind can be obtained in this manner.

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*) N_l denotes the Rees factor semigroup $N/(l)$ where $N=\{0, 1, 2, \cdots\}$ with addition and $(l)=\{x;\ x\geqq l\}$; hence "$x, y\in N$ and $x\equiv y$ (mod. N_l)" means "either $x\geqq l$ and $y\geqq l$ or $x=y<l$."

Remark on the earlier paper [1] In the 21st line, p. 39, [1], add

$$``P = \bigcup_{\lambda \in G} P_\lambda".$$

Reference

[1] T. Tamura: Notes on commutative archimedean semigroups. I. Proc. Japan Acad., **42**, 35-40 (1966).