120. *On the Leindler's Theorem*

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§ 1. Let f be an integrable function over $(0, 2\pi)$ and periodic with period 2π and let its Fourier series be

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

We write $\rho_n^2 = a_n^2 + b_n^2$. We shall give a simple proof of the following Leindler theorem (in a little stronger form) [1], [2], [3]:

Theorem 1. a) *If $\lambda(t)$ is a positive decreasing function on $(1, \infty)$, then*

$$(1) \qquad \sum_{n=2}^{\infty} \frac{1}{\lambda(n/2)} \Big(\sum_{m=n}^{\infty} \rho_m^q\Big)^{1/q}$$
$$\leqq A \int_0^1 \frac{dt}{t^2 \lambda(1/t)} \Big(\int_0^{2\pi} |f(x+2t) + f(x-2t) - 2f(x)|^p \, dx\Big)^{1/p},$$

where $1 < p \leqq 2$ and $1/p + 1/q = 1$.

b) *If $\lambda(t)$ is a positive increasing function on $(1, \infty)$, then* (1) *holds when $\lambda(n/2)$ is replaced by $\lambda(2n)$.*

c) *In the cases* a) *and* b), *the exponents $1/q$ and $1/p$ can be replaced by α/q and α/p where $0 < \alpha \leqq q$.*

By our method of proof of Theorem 1, we get another inequality similar to (1) in § 3.

§ 2. **Proof of Theorem 1.** We shall prove only the case a), since the remaining cases may be proved similarly. We have

$$(2) \qquad f(x+2t) + f(x-2t) - 2f(x) \sim 4 \sum_{n=1}^{\infty} \sin^2 nt (a_n \cos nx + b_n \sin nx)$$

and then, by the Hausdorff-Young theorem [4],

$$\Big(\sum_{k=1}^{\infty} \rho_k^q \sin^{2q} kt\Big)^{1/q} \leqq A \Big(\int_0^{2\pi} |f(x+2t) + f(x-2t) - 2f(x)|^p \, dx\Big)^{1/p}.$$

The right side of (1), except for a constant factor, is not less than

$$\int_0^1 \frac{dt}{t^2 \lambda(1/t)} \Big(\sum_{k=1}^{\infty} \rho_k^q \sin^{2q} kt\Big)^{1/q} = \int_1^{\infty} \frac{du}{\lambda(u)} \Big(\sum_{k=1}^{\infty} \rho_k^q \sin^{2q} \frac{k}{u}\Big)^{1/q}$$
$$\geqq \sum_{j=1}^{\infty} \frac{1}{\lambda(2^j)} \int_{2^j}^{2^{j+1}} \Big(\sum_{k=2^j}^{\infty} \rho_k^q \sin^{2q} \frac{k}{u}\Big)^{1/q} du$$
$$\geqq \sum_{j=1}^{\infty} \frac{1}{\lambda(2^j)} \Big\{\sum_{k=2^j}^{\infty} \Big(\int_{2^j}^{2^{j+1}} \Big(\rho_k^q \sin^{2q} \frac{k}{u}\Big)^{1/q} du\Big)^q\Big\}^{1/q}$$

by the Minkowski inequality. Since it is easy to see that there is a positive constant A such that

$$\int_{2^j}^{2^{j+1}} \sin^2 \frac{k}{u}\, du \geqq A(2^{j+1}-2^j)=A2^j \qquad \text{for all } k \geqq 2^j,$$

the above sum is not less than

$$A\sum_{j=1}^{\infty} \frac{1}{\lambda(2^j)}\left(\sum_{k=2^j}^{\infty} 2^{qj}\rho_k{}^q\right)^{1/q} = A\sum_{j=1}^{\infty} \frac{2^j}{\lambda(2^j)}\left(\sum_{k=2^j}^{\infty} \rho_k{}^q\right)^{1/q}$$
$$\geqq A\sum_{n=1}^{\infty} \frac{1}{\lambda(n/2)}\left(\sum_{m=n}^{\infty} \rho_m{}^q\right)^{1/q}.$$

Thus the theorem is proved.

§ 3. If we use Paley's theorem [4] for (2) instead of Hausdorff-Young's inequality, then

$$\left(\sum \rho_k{}^p k^{p-2} \sin^{2p} kt\right)^{1/p} \leqq A\left(\int_0^{2\pi} |f(x+2t)+f(x-2t)-2f(x)|^p\, dx\right)^{1/p}.$$

Using the method of proof of Theorem 1, we get

Theorem 2. a) *If $\lambda(t)$ is a positive decreasing function on* $(1, \infty)$, *then*

$$(3) \quad \sum_{n=2}^{\infty} \frac{1}{\lambda(n/2)}\left(\sum_{m=n}^{\infty} \rho_m{}^p m^{p-2}\right)^{1/p}$$
$$\leqq A\int_0^1 \frac{dt}{t^2\lambda(1/t)}\left(\int_0^{2\pi} |f(x+2t)+f(x-2t)-2f(x)|^p\, dx\right)^{1/p},$$

where $1<p\leqq 2$.

b) *If $\lambda(t)$ is a positive increasing function on* $(1, \infty)$, *then* (3) *holds when $\lambda(n/2)$ is replaced by $\lambda(2n)$.*

c) *In the cases* a) *and* b), *the exponent $1/p$ on the both sides of* (3) *may be replaced by α/p where $0<\alpha\leqq p$.*

References

[1] L. Leindler: Über verschiedene Konvergenzarten der trigonometrischen Reihen. Acta Sci. Math., **25**, 233-249 (1964).

[2] ——: Über verschiedene Konvergenzarten der trigonometrischen Reihen, II. Acta Sci. Math., **26**, 117-124 (1965).

[3] ——: Über Strukturbedingungen für Fourierreihen. Math. Zeits., **88**, 418-431 (1965).

[4] A. Zygmund: Trigonometric Series, II. Cambridge University Press, 101 and 120 (1959).