200. *Classification and the Characters of Irreducible Representations of $SU(p, 1)$*

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Let $SU(p, 1)$ be the matrix group defined by

$$g \in GL(p+1, c),\ \det g = 1,\ {}^t\bar{g}Jg = J, \tag{1}$$

where $J = \begin{bmatrix} 1_p & \\ & 1 \end{bmatrix}$ and 1_p is the $p \times p$ unit matrix.

In this note, we intend to classify all completely[1] (or more generally, quasi-simple[2]) irreducible representations of this group and to obtain explicite formulae of their characters.

§1. Sketch of classification of irreducible representations. It is convenient to handle, in place of $SU(p, 1)$, the group G which is defined by ${}^t\bar{g}Jg = J$ ($\det g = 1$ is not assumed). The results on $SU(p, 1)$ are easily deduced from the results on G.

A maximal compact subgroup U of G is the totality of matrices of the form $\begin{bmatrix} u & \\ & \lambda \end{bmatrix}$, $u \in U(p)$ and $|\lambda| = 1$. Every completely (or quasi-simple) irreducible representation, if it is restricted to U, contains any irreducible representation of U at most once [2], [6], [4, a, II].

To every quasi-simple irreducible representation, there exists a completely irreducible representation which is infinitesimally equivalent[2] to it. In the following, we say irreducible in place of quasi-simple irreducible for simplicity. There corresponds to every infinitesimally equivalent class of irreducible representations of G an algebraically equivalent class of algebraically irreducible representations of the Lie algebra $\mathfrak{G}$ of G [2], [4, a, I]. Therefore the classification of all infinitesimally equivalent classes of irreducible representations of G is reduced to the classification of algebraically irreducible representations of $\mathfrak{G}$ which is $\mathfrak{U}$-simple ($\mathfrak{U}$ is the Lie algebra of U), that is, any (finite dimensional) irreducible representation of $\mathfrak{U}$ is contained in it at most once.

Then we are lead to commutation relations between some difference operators in a vector space. The situation is quite similar to the case of the Lorentz group of n-th order [5, a, b], [6]. Using the results in [1], we obtain a system of difference equations. We can obtain all solutions of the difference equations, which satisfy some conditions expressing irreducibility of representations of $\mathfrak{G}$.

1) See [2]. 2) See [4, a, I].

This completes the classification.

Another method is with less calculations. We can infer typical solutions of the difference equations from the results [1] on finite dimensional representations of $\mathfrak{G}$. Thus we obtain typical representations of $\mathfrak{G}$, not necessarily irreducible. In stead of proving that the given solutions cover all needed solutions, we can reduce the problem to Theorem 4 in [4, a, II] as follows. For irreducible unitary representations of G of the principal continuous series, the corresponding representations of $\mathfrak{G}$ are contained in the typical representations obtained above. And the correspondence between the representations of G and the representations of $\mathfrak{G}$ are determined explicitely so that it can be analytically continued to the correspondence between the family of those representations of G that we construct as induced representations using Iwasawa decomposition [4, a, I] on the one hand and, on the other, the family of the typical representations of $\mathfrak{G}$. (Call them the elementary representations of G). The analytic continuation is with respect to some complex parameters of representations in each of the two families.

Generally, the elementary representations of G are not unitary nor irreducible. But, from the explicite forms of the corresponding representations of $\mathfrak{G}$, we can see which of them is irreducible or infinitesimally equivalent to a unitary representation. Moreover if it is not irreducible, it is semi-reducible and its irreducible components can be determined from infinitesimal standpoint (i.e. as irreducible representations of $\mathfrak{G}$). Thus the classification (and more) achieved [4, a, II]. This scheme of splitting elementary representations into (four or three) irreducible components is essential to calculate their characters.

§ 2. **Method of calculating characters.** To every infinitesimally equivalent class of quasi-simple irreducible representations, there corresponds the unique character [4, a, III]. The characters are determined completely by the values on Cartan subgroups of G (for regular elements). There exist in G exactly two Cartan subgroups H_0 and H_1 which are not conjugate by inner automorphisms of G, and they are of the following forms respectively:

$$H_0\colon h=\begin{bmatrix} e^{i\varphi_1} & & \\ & \ddots & \\ & & e^{i\varphi_{p+1}} \end{bmatrix},\quad i=\sqrt{-1}\quad\text{and}\quad \varphi_r\in\mathbf{R}, \tag{2}$$

$$H_1\colon h=\begin{bmatrix} e^{i\varphi_1} & & & & \\ & \ddots & & & \\ & & e^{i\varphi_{p-1}} & & \\ & & & e^{i\theta} & \\ & & & & e^{i\theta} \end{bmatrix}\begin{bmatrix} 1_{p-1} & & \\ & \operatorname{ch} t & \operatorname{sh} t \\ & \operatorname{sh} t & \operatorname{ch} t \end{bmatrix},\quad \varphi_r,\ \theta,\ t\in\mathbf{R}. \tag{2'}$$

The characters of elementary representations are known.[3] Therefore if an elementary representation is irreducible, we obtained the character of its equivalent class (Case I in § 3). If an elementary representation is reducible, we obtained a sum of characters of its irreducible components. The problem is reduced to solve these simultaneous linear equations. But the number of the unknowns are greater than the number of linearly independent equations.

For each irreducible representation of G, when it is restricted to U, we know the rule of splitting it into irreducible representations of U by the explicitely constructed representation of $\mathfrak{G}$ corresponding to it. Using this splitting rule and the Weyl's character formula for unitary groups, the value on H_0 of its character can be calculated. Therefore it remains only to calculate the value on H_1.

To an infinitesimal character there correspond some number of infinitesimally equivalent classes of irreducible representations. If an infinitesimal character is regular (Case II in § 3), we can distinguish square-integrable representations from the representations with the same infinitesimal character.

Originally the author did it as follows. The sums of characters of seemingly square-integrable representations with the same infinitesimal character can be calculated from the above linear equations. We can prove the Plancherel formula for G, using these sums and the characters of irreducible unitary representations of the principal continuous series [5, d]. As a collorary we can identify square-integrable representations. But after the recent work [4, c] of Harish-Chandra was published, we can identify square-integrable representations by the values on H_0 of their characters. Because, in the paper, there is given on a compact subgroup the values of the characters of square-integrable irreducible unitary representations. Incidentally, the values on H_0 of the characters of such irreducible representations that are irreducible components of non-irreducible elementary representations are different each other. (This situation for G and $SU(p, 1)$ is very special among simple Lie groups; see [5, c].)

Now the values on H_1 of the characters of a square-integrable representations can be determined uniquely by the fact that it defines on G a tempered invariant eigendistribution of all Laplace operators of G. Thus we can identify the square-integrable representations and obtain their characters.

With these informations, when the corresponding infinitesimal characters are regular, we can solve the above mentioned linear

3) See [4, b]. By the careful check, we see that the constant c in the formula of $T_{\lambda,\delta}$ in Theorem 2 of [4, b] (p. 511) is equal to the reciprocal of the order of the Weyl group of the compact Lie group $M_1^* = M_0 Z / D \cap Z$ (p. 497).

equations of characters of irreducible representations (Case II in § 3). When the infinitesimal characters of representations are singular, the author has not yet completed proof (Case III in § 3).

These results with detailed proofs will be published elsewhere. The author expresses his hearty thanks to Prof. H. Yoshizawa who has keen interest on these problems and constantly encouraged him.

§ 3. **Characters.** Now we give explicite formulae of the characters of irreducible representations. Put $\lambda_r = e^{i\varphi_r}$ $(1 \leqslant r \leqslant p+1)$ for H, and $\lambda_r = e^{i\varphi_r}$ $(1 \leqslant r \leqslant p-1)$, $\lambda_p = e^{i\theta - |t|}$, and $\lambda_{p+1} = e^{i\theta + |t|}$ for H_1 (if h is regular, $|\lambda_p| < 1 < |\lambda_{p+1}|$). We divide equivalent classes of irreducible representations of G into three parts.

(I) The first part is consisted of equivalent classes, each of which contains an irreducible elementary representation of G. The parameter of representations are a row of integers

$$\alpha = (l_1, l_2, \cdots, l_{p-1}) \text{ with } l_1 > l_2 > \cdots > l_{p-1},$$

and a pair of complex numbers (c_1, c_2) such that $c_1 + c_2 =$ an integer, and[4] neither c_1 nor c_2 are equal to an integer, or else, both c_1 and c_2 are equal to some of integers $l_1, l_2, \cdots, l_{p-1}$ resp. The character π of such a representation $\mathfrak{D}(\alpha; c_1, c_2)$ is as follows:

$\pi(h) = 0$ for $h \in H_0$;

$$\pi(h) = \frac{(-1)^{p+1}}{D_1(h)} |\lambda^{l_1}, \lambda^{l_2}, \cdots, \lambda^{l_{p-1}}| (\lambda_p^{c_1}\lambda_{p+1}^{c_2} + \lambda_p^{c_2}\lambda_{p+1}^{c_1}) \text{ for } h \in H_1, \tag{3}$$

where

$$D_1(h) = \prod_{1 \leqslant r < s \leqslant p+1} (\lambda_r - \lambda_s) \tag{4}$$

and

$$|\lambda^{l_1}, \lambda^{l_2}, \cdots, \lambda^{l_{p-1}}| = |\lambda^{l_1}, \lambda^{l_2}, \cdots, \lambda^{l_{p-1}}|_{\lambda=\lambda_1,\lambda_2,\cdots,\lambda_{p-1}} = \begin{vmatrix} \lambda_1^{l_1}, & \lambda_1^{l_2}, & \cdots, & \lambda_1^{l_{p-1}} \\ \lambda_2^{l_1}, & \lambda_2^{l_2}, & \cdots, & \lambda_2^{l_{p-1}} \\ \vdots & \vdots & & \vdots \\ \lambda_{p-1}^{l_1}, & \lambda_{p-1}^{l_2}, & \cdots, & \lambda_{p-1}^{l_{p-1}} \end{vmatrix}. \tag{5}$$

(II) The second part is consisted of those equivalent classes, each of which has a regular infinitesimal character and contains an irreducible component of a non-irreducible elementary representation. The parameter of representations are a row of integers $(l_0, l_1, l_2, \cdots, l_p)$ such that $l_0 > l_1 > l_2 > \cdots > l_p$,[5] and a pair of integers (i, j), $0 \leqslant i < j \leqslant p+1$.

The character of the representation $\mathrm{D}^{i,j}_{(l_0,l_1,\cdots,l_p)}$, is as follows:

$$\pi(h) = \frac{(-1)^{p+(j-i-1)}}{D_1(h)} \times \begin{vmatrix} \lambda^{l_0}, & \lambda^{l_1}, & \cdots, & \lambda^{l_{i-1}}, & \lambda^{l_i}, & \cdots, & \lambda^{l_{j-1}}, & \lambda^{l_j}, & \cdots, & \lambda^{l_p} \\ 0, & 0, & \cdots, & 0, & \lambda_{p+1}^{l_i}, & \cdots, & \lambda_{p+1}^{l_{j-1}}, & 0, & \cdots, & 0 \end{vmatrix}_{\lambda=\lambda_1,\cdots,\lambda_p} \text{ on } H_0 \tag{6}$$

4) The 2nd condition on (c_1, c_2) expresses that the elementary representation with the parameter $(\alpha; c_1, c_2)$ is irreducible.

$$
\pi(h)=\frac{(-1)^{p+(j-i-1)}}{D_1(h)}\left\{\begin{vmatrix} \lambda^{l_0}, \lambda^{l_1}, \cdots, \lambda^{l_{i-1}}, \lambda^{l_i}, & \lambda^{l_{i+1}}, \cdots, \lambda^{l_{j-1}}, \lambda^{l_j}, \cdots, \lambda^{l_p} \\ 0, 0, \cdots, 0, \quad \lambda_p^{l_i}, & \lambda_p^{l_{i+1}}, \cdots, \lambda_p^{l_{j-1}}, 0, \cdots, 0 \\ 0, 0, \cdots, 0, \quad \lambda_{p+1}^{l_i}, & \lambda_{p+1}^{l_{i+1}}, \cdots, \lambda_{p+1}^{l_{j-1}}, 0, \cdots, 0 \end{vmatrix}\right.
$$

$$
+\begin{vmatrix} \lambda^{l_0}, \cdots, \lambda^{l_{i-1}}, \lambda^{l_i}, \cdots, \lambda^{l_{j-1}}, \lambda^{l_j}, \cdots, \lambda^{l_p} \\ \lambda_p^{l_0}, \cdots, \lambda_p^{l_{i-1}}, 0, \cdots, 0, \quad 0, \cdots, 0 \\ 0, \cdots, 0, \quad \lambda_{p+1}^{l_i}, \cdots, \lambda_{p+1}^{l_{j-1}}, 0, \cdots, 0 \end{vmatrix}
$$

$$
\left.+\begin{vmatrix} \lambda^{l_0}, \cdots, \lambda^{l_{i-1}}, \lambda^{l_i}, \cdots, \lambda^{l_{j-1}}, \lambda^{l_j}, \cdots, \lambda^{l_p} \\ 0, \cdots, 0, \quad \lambda_p^{l_i}, \cdots, \lambda_p^{l_{j-1}}, 0, \cdots, 0 \\ 0, \cdots, 0, \quad 0, \cdots, 0, \quad \lambda_{p+1}^{l_j}, \cdots, \lambda_{p+1}^{l_p} \end{vmatrix}\right\}, \quad \text{on } H_1. \tag{7}
$$

Here the last symbol in (6) expresses the $(p+1)\times(p+1)$ determinant whose r-th row $(1\leqslant r\leqslant p)$ is $\lambda^{l_0}, \lambda^{l_1}, \cdots, \lambda^{l_p}$, and $(p+1)$-th row is $0, 0, \cdots, 0, \lambda_{p+1}^{l_i}, \lambda_{p+1}^{l_{i+1}}, \cdots, \lambda_{p+1}^{l_{j-1}}, 0, \cdots, 0$. The symbols in (7) expresses anologous determinants and these are an extention of. Weyl's notation (5).

The representation $D^{i,j}$ is square-integrable if and only if $j=i+1$, that is, the representations $D^{0,1}, D^{1,2}, \cdots, D^{p,p+1}$. The representations $D^{0,p+1}$ are finite dimensional. The representations $D^{0,1}$ and $D^{p,p+1}$ are the "type 1" irreducible representations and their contragradient ones are constructed by Graev on bounded domains and are square-integrable [3].

(III) The 3rd part is the limit case of the 2nd part, and is consisted of equivalent classes with singular infinitesimal characters, each of which contains an irreducible component of a non-irreducible elementary representation. The parameter of representations are a row of integers $(l_0, l_1, l_2, \cdots, l_p)$ such that

$$
l_0>l_1>\cdots>l_{i-1}=l_i>l_{i+1}>\cdots>l_p \quad (0\leqslant i\leqslant p) \tag{8}
$$

and a pair (i, j) or (j, i) according to $i<j\leqslant p$ or $0\leqslant j<i$ (with the same i). There exist[6] irreducible representations of G named as $D^{i,j}_{(l_0,l_1,\cdots,l_p)}$ if $i<j$, and $D^{j,i}_{(l_0,l_1,\cdots l_p)}$ if $j<i$. They are given explicitly as the corresponding representations of $\mathfrak{G}$ (i.e., in infinitesimal form).

Their characters are expressed by the formula (6) for $h\in H_0$ and, probably, by the formula (7) for $h\in H_1$. The values on H_0 of characters of equivalent classes of 3rd part are different each other, therefore we identify each irreducible representation by the value on H_0 of its character.

As mentioned in §2, the proof of the formula (7) for $h\in H_1$ is not complete. But, the author has proved the following facts:

(a) The tempered invariant eigendistribution, whose value on

5) These integers determine the infinitesimal character of the representation. There are $(p+2)(p+1)/2$ different irreducible representations which have the same infinitesimal character.

6) There are p irreducible representations which have the same infinitesimal character.

H_0 is equal to that of $\mathrm{D}^{i-1,i}$ or $\mathrm{D}^{i,i+1}$ (formula (6)), is unique and is given on H_1 by the formula (7) for $\mathrm{D}^{i-1,i}$ or $\mathrm{D}^{i,i+1}$ resp.

(b) Let i be the number in (8). If the formula (7) is valid for $\mathrm{D}^{i-1,i}$ or $\mathrm{D}^{i,i+1}$, then it is valid for all $\mathrm{D}^{i,j}$ and $\mathrm{D}^{j,i}$.

(c) For $\mathrm{D}^{0,1}$ and $\mathrm{D}^{p,p+1}$, (7) is valid. Because they are the representations constructed by Graev on bounded domains [3], by this explicite construction their characters can be calculated. They are not square-integrable but the limit case of squareintegrable representations. It is very probable that (7) is valid for any $\mathrm{D}^{i-1,i}$ and $\mathrm{D}^{i,i+1}$ in (III) as the limit cases of square-integrable representations in (II).

(d) For $p=2$ (i,e., $SU(2,1)$), (7) is valid. This is a immediate consequence of (b) and (c).

Finally, no invariant eigendistribution is linearly independent of the characters given in I, II, and III.

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