199. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. XXIV

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Theorem 66. For each value of $j=1, 2$, let $\{\lambda_\nu^{(j)}\}_{\nu=1,2,3,\dots}$ be a bounded infinite set of complex numbers; let D_j be a bounded, closed, and connected domain such that the closure $\overline{\{\lambda_\nu^{(j)}\}}$ has not any point in common with it; let N_j be a bounded normal operator whose point spectrum and continuous spectrum are given by $\{\lambda_\nu^{(j)}\}$ and $[\overline{\{\lambda_\nu^{(j)}\}}-\{\lambda_\nu^{(j)}\}]\cup D_j$ respectively (in fact, there exist such $N_j(j=1, 2)$ as we have already demonstrated); let

$$\chi_j(\lambda)=\sum_{\alpha=1}^{m_j}((\lambda I-N_j)^{-\alpha}h_{j\alpha}, g_j) \quad (\lambda\notin\overline{\{\lambda_\nu^{(j)}\}}\cup D_j,\ 1\leqq m_j\leqq\infty,\ j=1, 2),$$

where when $m_j<\infty$ $h_{j\alpha}$ and g_j are arbitrarily given elements in the complex abstract Hilbert space $\mathfrak{H}$ under consideration, whereas when $m_j=\infty$ $\{h_{j\alpha}\}_{\alpha\geqq 1}$ are so chosen as to satisfy the condition $\sum_{\alpha=1}^{\infty}\|(\lambda I-N_j)^{-1}\|^\alpha\|h_{j\alpha}\|<\infty$ for any $\lambda\notin\overline{\{\lambda_\nu^{(j)}\}}\cup D_j$ (this is possible); let $U_j(\lambda)=R_j(\lambda)+\chi_j(\lambda)$ where $R_j(\lambda)$ is an integral function; and let Γ be a rectifiable closed Jordan curve containing the sets $\overline{\{\lambda_\nu^{(1)}\}}\cup D_1$ and $\overline{\{\lambda_\nu^{(2)}\}}\cup D_2$ inside itself. Then

$$(54)\quad \frac{1}{2\pi i}\int_\Gamma U_1(\lambda)U_2(\lambda)d\lambda=\sum_{\alpha=1}^{m_1}\frac{(R_2^{(\alpha-1)}(N_1)h_{1\alpha}, g_1)}{(\alpha-1)!}+\sum_{\alpha=1}^{m_2}\frac{(R_1^{(\alpha-1)}(N_2)h_{2\alpha}, g_2)}{(\alpha-1)!}$$
$$(1\leqq m_j\leqq\infty,\ j=1, 2),$$

the complex line integral along Γ being taken counterclockwise; and moreover the two series on the right both are absolutely convergent when $m_j=\infty(j=1, 2)$. If, in addition to those hypotheses, there exists a rectifiable closed Jordan curve C such that $\overline{\{\lambda_\nu^{(1)}\}}\cup D_1$ lies inside C while $\overline{\{\lambda_\nu^{(2)}\}}\cup D_2$ lies outside C, then

$$(55)\quad \sum_{\alpha=1}^{m_1}\frac{(\chi_2^{(\alpha-1)}(N_1)h_{1\alpha}, g_1)}{(\alpha-1)!}+\sum_{\alpha=1}^{m_2}\frac{(\chi_1^{(\alpha-1)}(N_2)h_{2\alpha}, g_2)}{(\alpha-1)!}=0$$
$$(1\leqq m_j\leqq\infty,\ j=1, 2).$$

Proof. Since

$$\frac{1}{2\pi i}\int_\Gamma R_1(\lambda)R_2(\lambda)d\lambda=0$$

and since, as can be found from the Cauchy theorem and the expansions of $\chi_j\left(\frac{\rho}{\kappa}e^{i\theta}\right)(j=1, 2)$ shown in the preceding papers,

$$\frac{1}{2\pi i}\int_{\Gamma}\chi_1(\lambda)\chi_2(\lambda)d\lambda=0,$$

by making use of the complex spectral families $\{K_j(\lambda)\}$ of $N_j(j=1, 2)$ we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma}U_1(\lambda)U_2(\lambda)d\lambda&=\frac{1}{2\pi i}\int_{\Gamma}\chi_1(\lambda)R_2(\lambda)d\lambda+\frac{1}{2\pi i}\int_{\Gamma}\chi_2(\lambda)R_1(\lambda)d\lambda\\&=\frac{1}{2\pi i}\int_{\Gamma}\left\{\sum_{\alpha=1}^{m_1}\int_{\overline{\{\lambda_\nu^{(1)}\}}\cup D_1}\frac{1}{(\lambda-\zeta)^\alpha}d(K_1(\zeta)h_{1\alpha}, g_1)\right\}R_2(\lambda)d\lambda\\&+\frac{1}{2\pi i}\int_{\Gamma}\left\{\sum_{\alpha=1}^{m_2}\int_{\overline{\{\lambda_\nu^{(2)}\}}\cup D_2}\frac{1}{(\lambda-\zeta)^\alpha}d(K_2(\zeta)h_{2\alpha}, g_2)\right\}R_1(\lambda)d\lambda.\end{aligned}$$

Let d_j denote the distance between the two point sets Γ and $\overline{\{\lambda_\nu^{(j)}\}}\cup D_j$ for each value of $j=1, 2$. Then even if $m_j=\infty$, here the chain of inequalities

$$\sum_{\alpha=1}^{\infty}\left|\int_{\overline{\{\lambda_\nu^{(j)}\}}\cup D_j}\frac{1}{(\lambda-\zeta)^\alpha}d(K_j(\zeta)h_{j\alpha}, g_j)\right|\leqq\sum_{\alpha=1}^{\infty}\frac{\|h_{j\alpha}\|\,\|g_j\|}{d_j^\alpha}<\infty(\lambda\in\Gamma)$$

holds in accordance with the hypothesis $\sum_{\alpha=1}^{\infty}\|(\lambda I-N_j)^{-1}\|^\alpha\|h_{j\alpha}\|<\infty$ for $\lambda\notin\overline{\{\lambda_\nu^{(j)}\}}\cup D_j$. Since, in addition, $R_1(\lambda)$ and $R_2(\lambda)$ are both regular inside and on Γ, the final equality above is rewritten

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma}U_1(\lambda)U_2(\lambda)d\lambda&=\sum_{\alpha=1}^{m_1}\int_{\overline{\{\lambda_\nu^{(1)}\}}\cup D_1}\frac{R_2^{(\alpha-1)}(\zeta)}{(\alpha-1)!}d(K_1(\zeta)h_{1\alpha}, g_1)\\&+\sum_{\alpha=1}^{m_2}\int_{\overline{\{\lambda_\nu^{(2)}\}}\cup D_2}\frac{R_1^{(\alpha-1)}(\zeta)}{(\alpha-1)!}d(K_2(\zeta)h_{2\alpha}, g_2)\\&=\sum_{\alpha=1}^{m_1}\frac{(R_2^{(\alpha-1)}(N_1)h_{1\alpha}, g_1)}{(\alpha-1)!}+\sum_{\alpha=1}^{m_2}\frac{(R_1^{(\alpha-1)}(N_2)h_{2\alpha}, g_2)}{(\alpha-1)!}\\&\qquad(1\leqq m_j\leqq\infty,\ j=1, 2).\end{aligned}$$

If we now denote by L the length of Γ and set $M_j=\sup_{\lambda\in\Gamma}|R_j(\lambda)|$ for $j=1, 2$, then we here have

$$\sum_{\alpha=1}^{m_1}\frac{|(R_2^{(\alpha-1)}(N_1)h_{1\alpha}, g_1)|}{(\alpha-1)!}\leqq\frac{1}{2\pi}\sum_{\alpha=1}^{m_1}\frac{\|h_{1\alpha}\|\,\|g_1\|}{d_1^\alpha}M_2L<\infty\qquad(1\leqq m_1\leqq\infty)$$

and

$$\sum_{\alpha=1}^{m_2}\frac{|(R_1^{(\alpha-1)}(N_2)h_{2\alpha}, g_2)|}{(\alpha-1)!}\leqq\frac{1}{2\pi}\sum_{\alpha=1}^{m_2}\frac{\|h_{2\alpha}\|\,\|g_2\|}{d_2^\alpha}M_1L<\infty\qquad(1\leqq m_2\leqq\infty).$$

In consequence, the two series on the right of (54) converge absolutely for $m_1=m_2=\infty$.

Next we shall turn to the proof of the latter half of the theorem.

Since, by supposition, there exists a rectifiable closed Jordan curve C such that $\overline{\{\lambda_\nu^{(1)}\}}\cup D_1$ lies inside C and furthermore such that $\overline{\{\lambda_\nu^{(2)}\}}\cup D_2$ lies outside C, we denote its length by l. Then, from the fact that $\chi_2(\lambda)$ is regular inside and on C, we can find that

$$\frac{1}{2\pi i}\int_C \chi_1(\lambda)\chi_2(\lambda)d\lambda=\frac{1}{2\pi i}\int_C\Big\{\sum_{\alpha=1}^{m_1}\int_{\overline{\{\lambda_\nu^{(1)}\}}\cup D_1}\frac{1}{(\lambda-\zeta)^\alpha}d(K_1(\zeta)h_{1\alpha},\,g_1)\Big\}\chi_2(\lambda)d\lambda$$

$$=\sum_{\alpha=1}^{m_1}\int_{\overline{\{\lambda_\nu^{(1)}\}}\cup D_1}\frac{\chi_2^{(\alpha-1)}(\zeta)}{(\alpha-1)!}d(K_1(\zeta)h_{1\alpha},\,g_1)$$

$$=\sum_{\alpha=1}^{m_1}\frac{(\chi_2^{(\alpha-1)}(N_1)h_{1\alpha},\,g_1)}{(\alpha-1)!}\qquad(1\leqq m_1\leqq\infty);$$

and in addition, even if $m_1=\infty$

$$\sum_{\alpha=1}^{\infty}\frac{|(\chi_2^{(\alpha-1)}(N_1)h_{1\alpha},\,g_1)|}{(\alpha-1)!}$$

$$\leqq\frac{1}{2\pi}\sup_{\lambda\in C}|\chi_2(\lambda)|\sum_{\alpha=1}^{\infty}\sup_{\lambda\in C}||(\lambda I-N_1)^{-1}||^\alpha\,||h_{1\alpha}||\,||g_1||\,l<\infty.$$

On the other hand, since every $\zeta\in\overline{\{\lambda_\nu^{(2)}\}}\cup D_2$ lies outside C and since $\chi_1(\lambda)$ has its singularities within C, we can find from the course of the proof of Theorem 64 that

$$\frac{1}{2\pi i}\int_C \chi_1(\lambda)\chi_2(\lambda)d\lambda=\frac{1}{2\pi i}\int_C\chi_1(\lambda)\Big\{\sum_{\alpha=1}^{m_2}\int_{\overline{\{\lambda_\nu^{(2)}\}}\cup D_2}\frac{1}{(\lambda-\zeta)^\alpha}d(K_2(\zeta)h_{2\alpha},\,g_2)\Big\}d\lambda$$

$$=\sum_{\alpha=1}^{m_2}\int_{\overline{\{\lambda_\nu^{(2)}\}}\cup D_2}-\frac{\chi_1^{(\alpha-1)}(\zeta)}{(\alpha-1)!}d(K_2(\zeta)h_{2\alpha},\,g_2)$$

$$=-\sum_{\alpha=1}^{m_2}\frac{(\chi_1^{(\alpha-1)}(N_2)h_{2\alpha},\,g_2)}{(\alpha-1)!}\qquad(1\leqq m_2\leqq\infty)$$

according to the regularity of $\chi_1(\lambda)$ on the closed set $\overline{\{\lambda_\nu^{(2)}\}}\cup D_2$, and moreover the absolute convergency of the series on the right for $m_2=\infty$ is shown in the same manner as above.

The required result (55) is furnished by the two equalities just established. The theorem has thus been proved.

Remark. Let $M_j(\lambda)=||(\lambda I-N_j)^{-1}||$ for any fixed point $\lambda\notin\overline{\{\lambda_\nu^{(j)}\}}\cup D_j$, $(j=1,\,2)$; let $\{e_\nu\}_{\nu=1,2,3\dots}$ be a complete orthonormal set in $\mathfrak{H}$; and let

$$h_{j\alpha}=\frac{1}{\alpha!}\sum_{\nu=1}^{\infty}\frac{\kappa_\nu^{(j)}}{\sqrt{2^\nu}}e_\nu\in\mathfrak{H}\qquad(j=1,\,2),$$

where $\{\kappa_\nu^{(j)}\}_{\nu=1,2,3,\dots}$ is an infinite set of complex numbers such that $|\kappa_\nu^{(j)}|\leqq G_j<\infty(\nu=1,\,2,\,3,\,\cdots)$ for some positive constant G_j. Then we obtain

$$\sum_{\alpha=1}^{\infty}||(\lambda I-N_j)^{-1}||^\alpha\,||h_{j\alpha}||=\sum_{\alpha=1}^{\infty}M_j(\lambda)^\alpha\Big\{\sum_{\nu=1}^{\infty}\frac{|\kappa_\nu^{(j)}|^2}{2^\nu(\alpha!)^2}\Big\}^{\frac12}$$

$$\leqq\sum_{\alpha=1}^{\infty}M_j(\lambda)^\alpha\frac{G_j}{\alpha!}=G_j(e^{M_j(\lambda)}-1)<\infty,$$

so that $h_{j\alpha}(\in\mathfrak{H})$ can be so chosen as to satisfy the condition $\sum_{\alpha=1}^{\infty}||(\lambda I-N_j)^{-1}||^\alpha\,||h_{j\alpha}||<\infty$ for any $\lambda\notin\overline{\{\lambda_\nu^{(j)}\}}\cup D_j$.

Corollary 9. Let $\{\lambda_\nu^{(1)}\}_{\nu=1,2,3,\dots}$ and $\{\lambda_\nu^{(2)}\}_{\nu=1,2,3,\dots}$ both be bounded infinite sets of complex numbers such that their closures $\overline{\{\lambda_\nu^{(1)}\}}$ and

$\overline{\{\lambda_\nu^{(2)}\}}$ have no point in common; let $\{a_{j\alpha}^{(\nu)}\}_{\nu=1,2,3,\dots}$ and $\{b_j^{(\nu)}\}_{\nu=1,2,3,\dots}$ also be bounded infinite sets of complex numbers such that $\sum_{\nu=1}^{\infty}|a_{j\alpha}^{(\nu)}|^2<\infty$ $(j=1,2;\alpha=1,2,3,\cdots)$ and $\sum_{\nu=1}^{\infty}|b_j^{(\nu)}|^2<\infty$; and let

$$\chi_j(\lambda)=\sum_{\alpha=1}^{m_j}\sum_{\nu=1}^{\infty}\frac{a_{j\alpha}^{(\nu)}\overline{b_j^{(\nu)}}}{(\lambda-\lambda_\nu^{(j)})^\alpha}\quad(1\leqq m_j\leqq\infty,\ j=1,2),$$

where when $m_j=\infty$ $\{a_{j\alpha}^{(\nu)}\}_{\alpha,\nu\geqq1}$ are so chosen as to satisfy the condition

$$\sum_{\alpha=1}^{\infty}\sup_{\nu}|\lambda-\lambda_\nu^{(j)}|^{-\alpha}\left\{\sum_{\nu=1}^{\infty}|a_{j\alpha}^{(\nu)}|^2\right\}^{\frac{1}{2}}<\infty\quad(j=1,2)\tag{56}$$

for any $\lambda\notin\overline{\{\lambda_\nu^{(j)}\}}$. Then

$$\sum_{\alpha=1}^{m_1}\sum_{\nu=1}^{\infty}\frac{\chi_2^{(\alpha-1)}(\lambda_\nu^{(1)})a_{1\alpha}^{(\nu)}\overline{b_1^{(\nu)}}}{(\alpha-1)!}+\sum_{\alpha=1}^{m_2}\sum_{\nu=1}^{\infty}\frac{\chi_1^{(\alpha-1)}(\lambda_\nu^{(2)})a_{2\alpha}^{(\nu)}\overline{b_2^{(\nu)}}}{(\alpha-1)!}=0\tag{57}$$

$$(1\leqq m_j\leqq\infty,\ j=1,2),$$

where the two double series on the left both converge absolutely even if $m_1=m_2=\infty$.

Proof. For each value of $j=1,2$, let $\{e_\nu^{(j)}\}_{\nu=1,2,3,\dots}$ be a complete orthonormal system in $\mathfrak{H}$; let $\{e_{n_\nu}^{(j)}\}\subset\{e_\nu^{(j)}\}$ $(j=1,2)$; let N_j be a bounded normal operator in $\mathfrak{H}$ for each value of $j=1,2$ such that its point spectrum is given by $\{\lambda_\nu^{(j)}\}_{\nu=1,2,3,\dots}$ and furthermore such that $N_je_{n_\nu}^{(j)}=\lambda_\nu^{(j)}e_{n_\nu}^{(j)}(\nu=1,2,3,\cdots)$; and let $h_{j\alpha}=\sum_{\nu=1}^{\infty}a_{j\alpha}^{(\nu)}e_{n_\nu}^{(j)}\in\mathfrak{H}$ and $g_j=\sum_{\nu=1}^{\infty}b_j^{(\nu)}e_{n_\nu}^{(j)}\in\mathfrak{H}$ for $j=1,2$. If we denote by Δ_j the continuous spectrum of N_j for each value of $j=1,2$, then there is no difficulty in showing that

$$\begin{aligned}\sum_{\alpha=1}^{m_j}((\lambda I-N_j)^{-\alpha}h_{j\alpha},g_j)&=\sum_{\alpha=1}^{m_j}\int_{\{\lambda_\nu^{(j)}\}\cup\Delta_j}\frac{1}{(\lambda-\zeta)^\alpha}d(K_j(\zeta)h_{j\alpha},g_j)\\&=\sum_{\alpha=1}^{m_j}\int_{\{\lambda_\nu^{(j)}\}}\frac{1}{(\lambda-\zeta)^\alpha}d(K_j(\zeta)h_{j\alpha},g_j)\\&=\chi_j(\lambda)\qquad(1\leqq m_j\leqq\infty,\ \lambda\notin\overline{\{\lambda_\nu^{(j)}\}},\ j=1,2),\end{aligned}$$

and here it is obvious from the hypothesis (56) that, when $m_j=\infty$, $|\chi_j(\lambda)|\leqq\sum_{\alpha=1}^{\infty}\sum_{\nu=1}^{\infty}\left|\frac{a_{j\alpha}^{(\nu)}\overline{b_j^{(\nu)}}}{(\lambda-\lambda_\nu^{(j)})^\alpha}\right|<\infty$ for any $\lambda\notin\overline{\{\lambda_\nu^{(j)}\}}$ according to the Cauchy inequality. Hence the result (55) of Theorem 66 is applicable to the $\chi_j(\lambda)$ $(j=1,2)$. In addition, we have

$$\begin{aligned}\sum_{\alpha=1}^{m_1}\frac{(\chi_2^{(\alpha-1)}(N_1)h_{1\alpha},g_1)}{(\alpha-1)!}&=\sum_{\alpha=1}^{m_1}\int_{\{\lambda_\nu^{(1)}\}\cup\Delta_1}\frac{\chi_2^{(\alpha-1)}(\zeta)}{(\alpha-1)!}d(K_1(\zeta)h_{1\alpha},g_1)\quad(1\leqq m_1\leqq\infty)\\&=\sum_{\alpha=1}^{m_1}\int_{\{\lambda_\nu^{(1)}\}}\frac{\chi_2^{(\alpha-1)}(\zeta)}{(\alpha-1)!}d(K_1(\zeta)h_{1\alpha},g_1)\\&=\sum_{\alpha=1}^{m_1}\sum_{\nu=1}^{\infty}\frac{\chi_2^{(\alpha-1)}(\lambda_\nu^{(1)})a_{1\alpha}^{(\nu)}\overline{b_1^{(\nu)}}}{(\alpha-1)!}\end{aligned}$$

and similarly

$$\sum_{\alpha=1}^{m_2}\frac{(\chi_1^{(\alpha-1)}(N_2)h_{2\alpha},g_2)}{(\alpha-1)!}=\sum_{\alpha=1}^{m_2}\sum_{\nu=1}^{\infty}\frac{\chi_1^{(\alpha-1)}(\lambda_\nu^{(2)})a_{2\alpha}^{(\nu)}\overline{b_2^{(\nu)}}}{(\alpha-1)!}\quad(1\leqq m_2\leqq\infty).$$

By virtue of (55) these two equalities just established together imply the validity of the desired equality (57).

Next, let r be the distance between the two closed sets $\overline{\{\lambda_\nu^{(1)}\}}$ and $\overline{\{\lambda_\nu^{(2)}\}}$; let $C^{(\nu)}$ be the circle with center at $\lambda_\nu^{(1)}$ and radius $r/2$; and let $M_\nu = \sup_{\lambda\in C^{(\nu)}} |\chi_2(\lambda)|$. Since $\chi_2(\lambda)$ is regular at any point $\lambda \notin \overline{\{\lambda_\nu^{(2)}\}}$, M_ν ($\nu=1, 2, 3, \cdots$) are bounded and so there exists a positive constant M such that $M_\nu \leqq M$ ($\nu=1, 2, 3, \cdots$). As a result, Cauchy's inequality for the coefficients of the expansion of a regular function and the application of the maximum modulus principle to $\chi_2(\lambda)$ on the disc $\{\lambda: |\lambda-\lambda_\nu^{(1)}|\leqq r/2\}$ enable us to assert that

$$\frac{|\chi_2^{(\alpha-1)}(\lambda_\nu^{(1)})|}{(\alpha-1)!} \leqq \frac{M_\nu}{(r/2)^{\alpha-1}} \leqq M(2/r)^{\alpha-1} \quad (\alpha=1, 2, 3, \cdots),$$

so that

$$\sum_{\alpha=1}^{\infty}\sum_{\nu=1}^{\infty} \frac{|\chi_2^{(\alpha-1)}(\lambda_\nu^{(1)}) a_{1\alpha}^{(\nu)} \overline{b_1^{(\nu)}}|}{(\alpha-1)!} \leqq M\sum_{\alpha=1}^{\infty} (2/r)^{\alpha-1} \| h_{1\alpha} \| \| g_1 \| < \infty$$

by virtue of the hypothesis (56). Likewise we can verify the absolute convergency of the other double series on the left of (57).

The corollary has thus been proved.

Corollary 10. Let $\chi_j(\lambda)$ ($j=1, 2$) be the functions defined in the same manner as in Corollary 9, without using the foregoing hypothesis $\overline{\{\lambda_\nu^{(1)}\}} \cap \overline{\{\lambda_\nu^{(2)}\}} = \varnothing$; let $R_j(\lambda)$ ($j=1, 2$) be integral functions (inclusive of constants); let $U_j(\lambda) = R_j(\lambda) + \chi_j(\lambda)$ ($j=1, 2$), that is, let

$$U_j(\lambda) = R_j(\lambda) + \sum_{\alpha=1}^{m_j}\sum_{\nu=1}^{\infty} \frac{a_{j\alpha}^{(\nu)} \overline{b_j^{(\nu)}}}{(\lambda-\lambda_\nu^{(j)})^\alpha} \quad (1\leqq m_j \leqq \infty,\ j=1, 2),$$

where the coefficients $a_{j\alpha}^{(\nu)}$ and $b_j^{(\nu)}$ are subject to the conditions stated in Corollary 9; and let Γ be a rectifiable closed Jordan curve containing $\overline{\{\lambda_\nu^{(1)}\}} \cup \overline{\{\lambda_\nu^{(2)}\}}$ inside itself. Then

$$\frac{1}{2\pi i}\int_\Gamma U_1(\lambda)U_2(\lambda)d\lambda = \sum_{\alpha=1}^{m_1}\sum_{\nu=1}^{\infty} \frac{R_2^{(\alpha-1)}(\lambda_\nu^{(1)}) a_{1\alpha}^{(\nu)} \overline{b_1^{(\nu)}}}{(\alpha-1)!} + \sum_{\alpha=1}^{m_2}\sum_{\nu=1}^{\infty} \frac{R_1^{(\alpha-1)}(\lambda_\nu^{(2)}) a_{2\alpha}^{(\nu)} \overline{b_2^{(\nu)}}}{(\alpha-1)!}$$

$$(1\leqq m_j \leqq \infty,\ j=1, 2),$$

where the complex line integral on the left is extended counterclockwise around Γ; and moreover, the two double series on the right both converge absolutely even if $m_1 = m_2 = \infty$.

Proof. By means of (54) and the same reasoning as that used in the proof of Corollary 9, we can easily establish the present corollary.

Theorem 67. For each value of $j=1, 2$, let $U_j(\lambda)$ be the function defined in Theorem 66; let σ_j be the least positive constant subject to the condition that $\overline{\{\lambda_\nu^{(j)}\}} \cup D_j$ be on the disc $\{\lambda: |\lambda| \leqq \sigma_j\}$; let the expansion of $U_j(\lambda)$ on the exterior of this least disc be

$$U_j\Big(\frac{\rho}{\kappa}e^{i\theta}\Big) = \frac{1}{2}a_0^{(j)} + \frac{1}{2}\sum_{p=1}^{\infty} (a_p^{(j)} - ib_p^{(j)})\Big(\frac{e^{i\theta}}{\kappa}\Big)^p + \frac{1}{2}\sum_{p=1}^{\infty} (a_p^{(j)} + ib_p^{(j)})\Big(\frac{\kappa}{e^{i\theta}}\Big)^p$$

where $0<\kappa<1$, $\sigma_j<\rho<\infty$, and

$$\begin{cases} a_p^{(j)}=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} U_j(\rho e^{it}) \operatorname{cospt} dt \\ b_p^{(j)}=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} U_j(\rho e^{it}) \operatorname{sinpt} dt; \end{cases}$$

let $K_p^{(j)}=(a_p^{(j)})^2+(b_p^{(j)})^2$; and let Γ be the positively oriented curve defined in Theorem 66. Then $K_p^{(j)}$ $(p=1, 2, 3, \cdots)$ are constants independent of ρ for $j=1, 2$; and assuming that $\dfrac{K_{p+1}^{(j)}}{R_j^{(p+1)}(0)}$ denotes $\dfrac{1}{2\pi i}\displaystyle\int_\Gamma U_j(\lambda)\lambda^p d\lambda\times\dfrac{4}{(p+1)!}$ when $R_j^{(p+1)}(0)=0$, the equalities

$$\sum_{\alpha=1}^{m_1}\frac{(R_2^{(\alpha-1)}(N_1)h_{1\alpha},\ g_1)}{(\alpha-1)!}=\frac{1}{4}\sum_{p=0}^{\infty}(p+1)R_2^{(p)}(0)\frac{K_{p+1}^{(1)}}{R_1^{(p+1)}(0)} \quad (1\leqq m_1\leqq\infty) \tag{58}$$

and

$$\sum_{\alpha=1}^{m_2}\frac{(R_1^{(\alpha-1)}(N_2)h_{2\alpha},\ g_2)}{(\alpha-1)!}=\frac{1}{4}\sum_{p=0}^{\infty}(p+1)R_1^{(p)}(0)\frac{K_{p+1}^{(2)}}{R_2^{(p+1)}(0)} \quad (1\leqq m_2\leqq\infty) \tag{59}$$

hold for the respective ordinary parts $R_1(\lambda)$ and $R_2(\lambda)$ of $U_1(\lambda)$ and $U_2(\lambda)$.

Proof. Since it is apparent that the results of Theorem 65 are also valid for $U_j(\lambda)$ $(j=1, 2)$,

$$\frac{1}{2\pi i}\int_\Gamma \chi_1(\lambda)R_2(\lambda)d\lambda=\frac{1}{4}\sum_{p=0}^{\infty}(p+1)R_2^{(p)}(0)\frac{K_{p+1}^{(1)}}{R_1^{(p+1)}(0)}$$

and the series on the right is absolutely convergent. On the other hand, we have

$$\frac{1}{2\pi i}\int_\Gamma \chi_1(\lambda)R_2(\lambda)d\lambda=\sum_{\alpha=1}^{m_1}\frac{(R_2^{(\alpha-1)}(N_1)h_{1\alpha},\ g_1)}{(\alpha-1)!} \quad (1\leqq m_1\leqq\infty),$$

as will be seen from the course of the proof of Theorem 66. These equalities yield the required relation (58). In a similar manner, the relation (59) can be established. Each of $K_p^{(j)}$ $(p=1, 2, 3, \cdots)$ is of course a constant independent of ρ provided that $\sigma_j<\rho<\infty$ $(j=1, 2)$, as we have already pointed out in Theorem 65.

With these results, the proof of the theorem is complete.