227. *On an Addition Theorem for M-Spaces*

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1. **Introduction.** Let X be a topological space and $\{A_\alpha\}$ a locally finite closed covering of X. As is well known, if each subspace A_α has one of the following properties, then the whole space X has also the same property (see, for instance, K. Morita [4] and J. Nagata [8]):

(a) being a normal space, (b) being countably paracompact,
(c) being collectionwise normal, (d) being perfectly normal,
(e) being paracompact and normal, (f) being metrizable.

Recently, K. Morita [7] has introduced the notion of M-spaces. He calls a topological space X an *M-space* if there exists a normal sequence $\{\mathfrak{U}_n \mid n=1, 2, \cdots\}$ of open coverings of X satisfying the condition (*) below:

(*) If a family $\mathfrak{K}$ consisting of a countable number of subsets of X has the finite intersection property and containing as its member a subset of $\mathrm{St}(x_0, \mathfrak{U}_n)$ for each n and for some fixed point x_0 in X, then $\bigcap\{\bar{K} \mid K \in \mathfrak{K}\} \neq \emptyset$.

In this note, we shall establish an analogous result for the notion of M-spaces; namely, we shall prove the following theorem:

Theorem 1. *Let $\{A_\alpha\}$ be a locally finite covering of a Hausdorff space X and each A_α be a closed G_δ-subset of X. If each A_α is a normal M-space with respect to its relative topology, then the whole space X is also a normal M-space.*

The next § 2 is devoted to the proof of this theorem, and in § 3 we shall deduce some of its immediate consequences. Most terminologies and notations used in this note are the same as those of J. W. Tukey [12].

Finally, I wish to express my hearty thanks to Prof. K. Morita who has given me many kindful suggestions and advices.

2. **Proof of Theorem 1.** Our proof of Theorem 1 rests upon the following two lemmas.

Lemma 1. *Let $\{\mathfrak{G}_n \mid n=1, 2, \cdots\}$ be a normal sequence of open coverings of a topological space X. Then there is another normal sequence $\{\mathfrak{H}_n \mid n=1, 2, \cdots\}$ of open coverings of X having the properties:*

(i) *Each $\mathfrak{H}_n$ is a refinement of $\mathfrak{G}_n$.*

(ii) *Each* $\mathfrak{H}_n$ *is locally finite.*

Lemma 2. *Let* $\{F_\alpha \mid \alpha \in \Omega\}$ *be a family of closed subsets in a space* X *and for each* $\alpha \in \Omega$, *let* $\mathfrak{G}_\alpha$ *be a locally finite family of open subsets in* X *such that* $F_\alpha \subset H_\alpha$, *where* $H_\alpha = \bigcup\{G \mid G \in \mathfrak{G}_\alpha\}$. *Denote by* $\tilde{\mathfrak{G}}_\alpha$ *the open covering of* X *consisting of the set* $X - F_\alpha$ *and the sets in* $\mathfrak{G}_\alpha$. *If the family* $\{H_\alpha \mid \alpha \in \Omega\}$ *is locally finite, then the family* $\bigwedge\{\tilde{\mathfrak{G}}_\alpha \mid \alpha \in \Omega\}$[1] *is a locally finite open covering of* X.

Lemma 1 is an easy consequence of Theorem 1.2 in [5] and the proof of Lemma 2 is similar to that of Theorem 1.2 in [3].

Now we proceed to prove Theorem 1. Let $\{A_\alpha \mid \alpha \in \Omega\}$ be a locally finite closed covering of a Hausdorff space X and suppose that each A_α is a normal M-space with respect to its relative topology. According to a result of A. Okuyama [10], each A_α is collectionwise normal and countably paracompact, and hence the whole space X is also collectionwise normal and countably paracompact (see K. Morita [4]). Consequently, from a theorem of M. Katětov [2] we see that there exists a locally finite open family $\{H_\alpha \mid \alpha \in \Omega\}$ of open subsets of X such that $H_\alpha \supset A_\alpha$ for each $\alpha \in \Omega$ (see also V. Šedivá [11]). Considering the G_δ-property of A_α, we can find a countable family $\{G_n^{(\alpha)} \mid n = 1, 2, \cdots\}$ of open subsets of X such that $\bigcap_{n=1}^{\infty} G_n^{(\alpha)} = A_\alpha$ and that $G_n^{(\alpha)} \subset H_\alpha$ and $G_{n+1}^{(\alpha)} \subset G_n^{(\alpha)}$ for each n.

Since each A_α is an M-space, there exists a normal sequence $\{\mathfrak{U}_n^{(\alpha)} \mid n = 1, 2, \cdots\}$ of open coverings of the subspace A_α satisfying the condition (*) with respect to the subspace A_α. Moreover, we can assume that each covering $\mathfrak{U}_n^{(\alpha)}$ is locally finite in A_α by Lemma 1. Applying a theorem of C. H. Dowker [1] to this covering, we obtain a locally finite open covering $\mathfrak{M}_n^{(\alpha)}$ of X such that $\mathfrak{M}_n^{(\alpha)} \cap A_\alpha$[2] is a refinement of $\mathfrak{U}_n^{(\alpha)}$. Denote by $\tilde{\mathfrak{M}}_n^{(\alpha)}$ the open covering of the space X formed by $X - A_\alpha$ and the sets of the form $G_n^{(\alpha)} \cap M$, where $M \in \mathfrak{M}_n^{(\alpha)}$, and construct the covering

$$(1) \qquad \bigwedge\{\tilde{\mathfrak{M}}_n^{(\alpha)} \mid \alpha \in \Omega\} = \mathfrak{N}_n, \qquad (n = 1, 2, \cdots).$$

Then we see from Lemma 2 that $\mathfrak{N}_n$ is a locally finite open covering of the space X.

Now we shall construct by induction a normal sequence $\{\mathfrak{B}_n\}$ of open coverings of X in the following way. First, we put $\mathfrak{B}_1 = \mathfrak{N}_1$. Let $n > 1$ and suppose that the locally finite open coverings $\mathfrak{B}_1 (i = 1, 2, \cdots, n-1)$ have been defined in such a way that (i) $\mathfrak{B}_i$ is a refinement of $\mathfrak{N}_i$, and (ii) $\mathfrak{B}_i$ is a star-refinement of $\mathfrak{B}_{i-1}$. Since $\mathfrak{B}_{n-1}$ is a locally finite open covering of the normal space X, it has a locally finite open

1) As to this notation, see J. W. Tukey [12].

2) Here $\mathfrak{M}_n^{(\alpha)} \cap A_\alpha$ means a family $\{M \cap A_\alpha \mid M \in \mathfrak{M}_n^{(\alpha)}\}$.

star-refinement $\mathfrak{P}_n$. Let $\mathfrak{N}_n \wedge \mathfrak{P}_n = \mathfrak{V}_n$. Then the above relations (i) and (ii) are also valid for $i=n$.

This sequence $\{\mathfrak{V}_n\}$ is obviously a normal sequence of open coverings of X. It remains to prove that the sequence $\{\mathfrak{V}_n\}$ satisfies the M-space condition (*). To prove this, let $\mathfrak{K}=\{K_n \mid n=1, 2, \cdots\}$ be any family consisting of a countable number of subsets of X having the finite intersection property and suppose that $\mathfrak{K}$ contains as its member a subset K_{n_i} of $\mathrm{St}(x_0, \mathfrak{V}_i)$ for every i and for some fixed point x_0 of X. We have to show $\bigcap\{\bar{K} \mid K \in \mathfrak{K}\} \neq \emptyset$. Let $\Delta(x_0)=\{\alpha \in \Omega \mid x_0 \in H_\alpha\}$ and $\Delta_0(x_0)=\{\alpha \in \Omega \mid x_0 \in A_\alpha\}$. Then the index-sets $\Delta(x_0)$ and $\Delta_0(x_0)$ are finite subsets of Ω by virtue of local finiteness of the family $\{H_\alpha \mid \alpha \in \Omega\}$, and the set $\Delta_0(x_0)$ is contained in $\Delta(x_0)$.

First, we shall prove that for every i we have:

(2) $$\mathrm{St}(x_0, \mathfrak{V}_i) \subset \mathrm{St}(x_0, \mathfrak{N}_i) \subset \bigcup\{A_\alpha \mid \alpha \in \Delta(x_0)\}.$$

Indeed, let N be any member of $\mathfrak{N}_i$ containing the point x_0. The set N has the form

(3) $$N=\bigcap\{X-A_\alpha \mid \alpha \in \Omega_1\} \cap (\bigcap\{G_i^{(\alpha)} \cap U_\alpha \mid \alpha \in \Omega_2\}),$$

where U_α is some set belonging to $\mathfrak{M}_i^{(\alpha)}$, and Ω_1 and Ω_2 are disjoint subsets of Ω such that $\Omega_1 \cup \Omega_2 = \Omega$. If β is any index not contained in $\Delta(x_0)$, then β must be contained in Ω_1, whence we have $N \subset X-A_\beta$. Since this relation holds for any $\beta \notin \Delta(x_0)$, it follows that

$$N \subset \bigcap\{X-A_\beta \mid \beta \notin \Delta(x_0)\} = X - \bigcup\{A_\beta \mid \beta \notin \Delta(x_0)\} \subset \bigcup\{A_\alpha \mid \alpha \in \Delta(x_0)\}.$$

Hence the relation (2) is proved.

Next, we shall show that there is some integer i_0 such that $i \geqq i_0$ implies

(4) $$\mathrm{St}(x_0, \mathfrak{V}_i) \subset \bigcup\{A_\alpha \mid \alpha \in \Delta_0(x_0)\}.$$

Indeed, in case $\Delta(x_0)=\Delta_0(x_0)$, we have only to choose $i_0=1$. Otherwise, we choose any $\beta \in \Delta(x_0)-\Delta_0(x_0)$. Then $x_0 \notin A_\beta$ and hence $x_0 \notin G_i^{(\beta)}$ for some $i=i(\beta)$. Therefore, for any $j \geqq i(\beta)$, we have

$$\mathrm{St}(x_0, \mathfrak{V}_j) \subset \mathrm{St}(x_0, \mathfrak{V}_{i(\beta)}) \subset \mathrm{St}(x_0, \widetilde{\mathfrak{M}}_{i(\beta)}^{(\beta)}) \subset X-A_\beta.$$

Consequently, if we choose $i_0 = \mathrm{Max}\,\{i(\beta) \mid \beta \in \Delta(x_0)-\Delta_0(x_0)\}$, we see that the relation (4) is valid for this i_0.

Finally, we shall prove that the family $\mathfrak{K} \cap A_{\alpha_0}$ has the finite intersection property for some $\alpha_0 \in \Delta_0(x_0)$. Let us assume the contrary. Then for each $\alpha \in \Delta_0(x_0)$, we can find a finite set Γ_α of natural numbers such that $\bigcap\{K_i \cap A_\alpha \mid i \in \Gamma_\alpha\} = \emptyset$. Let $\Gamma = \bigcup\{\Gamma_\alpha \mid \alpha \in \Delta_0(x_0)\}$. Then we have from (4),

$$\begin{aligned} K_{n_{i_0}} \cap (\bigcap\{K_i \mid i \in \Gamma\}) &= K_{n_{i_0}} \cap (\bigcap\{A_\alpha \mid \alpha \in \Delta_0(x_0)\}) \cap (\bigcap\{K_i \mid i \in \Gamma\}) \\ &= K_{n_{i_0}} \cap [\bigcap_{\alpha \in \Delta_0(x_0)} (A_\alpha \cap (\bigcap_{i \in \Gamma} K_i))] = \emptyset. \end{aligned}$$

This contradicts the finite intersection property of the family $\mathfrak{K}$. Hence, for some $\alpha_0 \in \Delta_0(x_0)$, the family $\mathfrak{K} \cap A_{\alpha_0}$ has the finite intersection property, and furthermore, for every $i \geqq i_0$, we have

$$
\begin{aligned}
K_{n_i} \cap A_{\alpha_0} &\subset \mathrm{St}(x_0, \mathfrak{V}_i) \cap A_{\alpha_0} \subset \mathrm{St}(x_0, \mathfrak{N}_i) \cap A_{\alpha_0} \\
&\subset \mathrm{St}(x_0, \tilde{\mathfrak{M}}_i^{(\alpha_0)}) \cap A_{\alpha_0} \subset \mathrm{St}(x_0, \mathfrak{M}_i^{(\alpha_0)}) \cap A_{\alpha_0} \\
&\subset \mathrm{St}(x_0, \mathfrak{U}_i^{(\alpha_0)}),
\end{aligned}
$$

because $x_0 \in A_{\alpha_0}$. Since the subspace A_{α_0} is an M-space, we have

$$\bigcap \{\overline{K \cap A_{\alpha_0}} \cap A_{\alpha_0} \mid K \in \mathfrak{K}\} \neq \emptyset,$$

whence we infer that $\bigcap \{\bar{K} \mid K \in \mathfrak{K}\} \neq \emptyset$. Thus, the proof of Theorem 1 is completed.

The problem whether Theorem 1 is valid without the G_δ-condition of A_α remains open.

3. Some applications. Finally, we shall deduce some results from Theorem 1.

Theorem 2. *Let X be a paracompact Hausdorff space which is locally an M-space, i.e., each point x in X has a neighborhood U_x such that U_x is an M-space. Then the space X itself is an M-space.*

Proof. By paracompactness of the space X, the open covering $\{\mathrm{Int}\, U_x \mid x \in X\}$ has a locally finite open refinement $\{G_\alpha \mid \alpha \in \Omega\}$, which also has an open refinement $\{H_\alpha \mid \alpha \in \Omega\}$ such that $G_\alpha \supset \bar{H}_\alpha$ for each $\alpha \in \Omega$. By normality of the space X, we can find closed G_δ-subsets A_α such that $\bar{H}_\alpha \subset A_\alpha \subset G_\alpha$ for each α. Then each A_α is a closed subspace of some M-space U_x and hence is an M-space and moreover, a normal space. Thus, the theorem follows from the above Theorem 1.

Theorem 3. *Let $\{G_\alpha\}$ be a σ-locally finite covering of a normal Hausdorff space X and suppose that each G_α is an open F_σ-subset of X. If each G_α is an M-space, then so also is X.*

Proof. According to K. Morita [7, Theorem 1.2], $\{G_\alpha\}$ is a normal covering of X and hence has a locally finite refinement. Therefore, by the similar method as above, we can find a locally finite refinement $\{F_\beta\}$ of $\{G_\alpha\}$ such that each F_β is a closed G_δ-subset of X. Thus, the theorem is an immediate consequence of Theorem 1.

Theorem 4 (J. Nagata and Yu. Smirnov). *Let $\{A_\alpha\}$ be a locally finite closed covering of a topological space X. If each A_α is metrizable, then so also is X.*

Proof. Since each A_α is paracompact and perfectly normal, from [4] we see that X is also paracompact and perfectly normal. Consequently, X is an M-space by Theorem 1. On the other hand, the product space $X \times X$ is perfectly normal because it is a union of the locally finite family of metrizable and hence perfectly normal subspaces $A_\alpha \times A_\beta$. Applying A. Okuyama's metrization theorem [9], we conclude that X is metrizable.

Theorem 5. *Let $\{A_n \mid n=1, 2, \cdots\}$ be a countable family of closed subsets of a space X having the property:*

(**) $$\bigcup\{\operatorname{Int} A_n \mid n=1, 2, \cdots\}=X.$$

If each A_n is a normal M-space, then so also is X.

Proof. Let $C_1=A_1$ and $C_n=A_n-\bigcup_{i<n}\operatorname{Int} A_i$ for $n>1$. It is clear that $\{C_n \mid n=1, 2, \cdots\}$ is a locally finite closed covering of X and that each C_n is a normal M-space and hence a countably paracompact space (see [7]). Therefore X is also countably paracompact and normal (cf. [4]) and consequently, we can find a locally finite family $\{D_n\}$ such that $C_n\subset D_n\subset \operatorname{Int} A_n$ and each D_n is a closed G_δ-subset of X. Thus this theorem also follows from Theorem 1.

If we drop the condition (**) in Theorem 5, the theorem fails to hold. There exists a non-M-space which is a countable union of closed M-subspaces; for instance, any non-metrizable CW-complex is such a space (see Morita [6]).

Corollary 1. *If, in the above Theorem 5, each A_n is metrizable, then so also is the space X.*

The proof is similar to the above; we use Theorem 4 instead of Theorem 1.

Corollary 2. *Let A be an open F_σ-subset of a normal Hausdorff M-space X. Then A is also a normal M-space.*

Proof. Let $A=\bigcup_n F_n$, where each A_n is a closed subset of X. Then we can easily find closed subsets C_n in such a way that

$$F_n\subset \operatorname{Int} C_n\subset C_n\subset A.$$

From this we have $A=\bigcup_n \operatorname{Int} C_n$, and the theorem follows from Theorem 5.

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