226. Remark on Eigenfunctions of the Operators $-\Delta+(qx)$

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Introduction. It is stated in [1] that if the operator $L\equiv-\Delta+q(x)$, where Δ is 3-dimension Laplacian and $q(x)$ is sufficiently differentiable real-valued function with compact support in 3-dimension Euclidean space R^3, has no eigenvalue, then solution $u(x)$ of equation $Lu=-\lambda^2 u$ where λ is a complex number satisfying $Re\lambda\geqq 0$ equals to zero identically if $u(x)$ is a twice continuously differentiable function and also $u(x)=O(|x|^{-1})$ as $|x|\to\infty$.

In §1 we give an example such that L has no eigenvalue, but that for $\lambda=0$, $Lu=\lambda u$ has a solution, not zero identically which is not an eigenfunction, but $u(x)=O(|x|^{-1})$ as $|x|\to\infty$, where $q(x)$ has a compact support and for any positive number ε

$$-q(x)\leqq\left(\frac{1}{4}+\varepsilon\right)\frac{1}{|x|^2}$$

and also there exist some r_1, r_2 $(0<r_1<r_2<\infty)$ and for $r_1\leqq|x|\leqq r_2$

$$-q(x)\not\leqq\frac{1}{4}\frac{1}{|x|^2}.$$

From this example, we can construct a solution of wave equation such that $\frac{\partial^2 u}{\partial t^2}-\Delta u+qu=0$ for $t>0$, its initial data $u(0,x)$ and $\frac{\partial u}{\partial t}(0,x)$ have compact supports resprectively, but that $\lim_{t\to\infty}u(t,x)$ does not vanish for any $x\in R^3$.

Our considerations of the method were suggested by those of the method used in [2]. Next we give its proof in §2, and consider the influence which $\frac{1}{4}\frac{1}{|x|^2}$ has on the spectrum of L in §3.

§1. We consider a differential operator $L\equiv-\Delta+q(x)$ defined on R^3, where $q(x)$ is a twice continuously differentiable real-valued function and also $q(x)=O(|x|^{-2-h})$ $(h>0)$ as $|x|\to\infty$. On this case L has a unique self-adjoint extension on $L^2(R^3)$ and its domain is the set of all functions whose partial derivatives of order$\leqq 2$ in distribution sense belong to $L^2(R^3)$. We also denote the extended operator by L. Furthermore we write $|x|=r$.

Example 1. We set

$$q(x)=\begin{cases}-k^2 & \text{for } 0\leqq r<r_1,\\ -\left(\frac{1}{4}+\varepsilon\right)\frac{1}{r^2} & \text{for } r_1\leqq r\leqq r_2,\\ 0 & \text{for } r>r_2,\end{cases}$$

and

$$w(r)=\begin{cases}\sin kr & \text{for } 0\leqq r<r_1,\\ c_1 r^{\frac{1}{2}}\sin(\sqrt{\varepsilon}\log r) & \text{for } r_1\leqq r_2\leqq r,\\ c_2 & \text{for } r>r_2,\end{cases}$$

where ε is an arbitrary positive number, and k, r_1, r_2, c_1, c_2 will be determined later such that $w(r)$ is positive on $(0, \infty)$ and continuously differentiable on $(0, \infty)$. Next we set

$$u(x)=r^{-1}w(r)$$

and also we write

$$\tilde{u}(x)=u^*\varphi_\delta(x),$$
$$\tilde{q}(x)=\frac{(qu)^*\varphi_\delta(x)}{u^*\varphi_\delta(x)},$$

where

$$\varphi_\delta(x)=\delta^{-3}\varphi\left(\frac{r}{\delta}\right),$$
$$\varphi(r)\in C^\infty([0, \infty)),$$
$$\varphi(r)\geqq 0 \quad \text{for } r\in[0, \infty),$$
$$\varphi(r)=0 \quad \text{for } r\geqq 1,$$
$$\varphi(r)=1 \quad \text{for } 0\leqq r\leqq \tfrac{1}{2},$$
$$\int_0^\infty \varphi(r)dx=1,$$

and δ is a sufficiently small positive number. Then $u^*\varphi_\delta(x)\neq 0$, so $\tilde{u}(x)\in C^\infty(R^3)$, $\tilde{q}(x)\in C^\infty(R^3)$ and $\tilde{u}$ satisfies an equation $-\Delta\tilde{u}+\tilde{q}\tilde{u}=0$, and also $r\tilde{u}\to c_2$ as $r\to\infty$. But $c_2>0$, so $\tilde{u}(x)$ does not belong to $L^2(R^3)$.

Now we divide $q(x)$ into $q(x)=q_+(x)-q_-(x)$, $q_+(x)\geqq 0$, $q_-(x)\geqq 0$.

Here to explain our significance of Example 1, we give two lemmas.

Lemma 1. *Let $q_-(x)\leqq\frac{1}{4}\frac{1}{r^2}$. Then the solution $u(x)$ of the equation $Lu=0$ equals to zero identically if for some $\varepsilon>0$, $u=O(r^{-\frac{1}{2}-\varepsilon})$ and $\frac{\partial u}{\partial x_i}=O(r^{-\frac{3}{2}-\varepsilon})$, $(i=1, 2, 3)$ as $r\to\infty$.*

Lemma 2. *Let $q(x)$ be a function which satisfies the following properties:*

i) $q(x)=q(r)$,

ii) *there exists a number $r_1(>0)$ such that $q(r)=0$ for $r\geqq r_1$,*

iii) $q_-\leqq(2+\frac{1}{4})\frac{1}{r^2}$.

Furthermore let $Lu=0$ *have a solution* u *such that* $u=u(r)>0$ *on* $(0, \infty)$, *then the operator* L *has no eigenvalue.*

§ 2. **1. The construction of w and of in Example 1.** We at first choose r_2 such that $0<r_2<1$ and $w'(r_2)=0$, that is,

$$\tan(\sqrt{\varepsilon}\log r_2)=-2\sqrt{\varepsilon}.$$

Next we choose k, r_1, c_1 such that

$$k>0,$$
$$0<r_1<r_2,$$
$$\sin kr_1=c_1 r_1^{\frac{1}{2}}\sin(\sqrt{\varepsilon}\log r_1),$$
$$k\cos kr_1=c_1 r_1^{-\frac{1}{2}}\{\tfrac{1}{2}\sin(\sqrt{\varepsilon}\log r_1)+\sqrt{\varepsilon}\cos(\sqrt{\varepsilon}\log r_1)\},$$
$$k^2\leqq(\tfrac{1}{4}+\varepsilon)\frac{1}{r_1^2},$$

and $c_1\sin(\sqrt{\varepsilon}\log r)>0$ for $r\in[r_1, r_2]$.
Furthermore we set

$$c_2=c_1 r_2^{\frac{1}{2}}\sin(\sqrt{\varepsilon}\log r_2)\qquad(>0).$$

Then $q(r)$ and $w(r)$ described in § 1 satisfy the relation

$$w''(r)=q(r)w(r).$$

We now set

$$u(x)=r^{-1}w(r),$$

then $u(x)>0$ and $ru(x)\to c_2$ as $r\to\infty$. From the above relation we see that $-\Delta u+qu=0$, but that $u\notin L^2(R^3)$.

2. Proof of Lemma 1. It is well known that if $u(x)\in C^2(R^3)$, and $u(x)=O(r^{-\frac{1}{2}-\varepsilon})$, $\dfrac{\partial u}{\partial x_i}=O(r^{-\frac{3}{2}-\varepsilon})$ $(i=1, 2, 3)$, $\varepsilon>0$, as $r\to\infty$, we then have the following inequality:

$$\int_{R^3}\frac{1}{4}\frac{|u(x)|^2}{r^2}dx\leqq\int_{R^3}|\operatorname{grad} u(x)|^2dx.$$

Both sides are equal if and only if $u(x)\equiv 0$. Now we prove Lemma 1. If $u(x)\not\equiv 0$, then from the assumption

$$\begin{aligned}0&=\int_{R^3}(-\Delta u+qu)\bar{u}dx\\&=\int_{R^3}(|\operatorname{grad} u|^2+q|u|^2)dx\\&>\int_{R^3}\left(\frac{1}{4r^2}-q_-\right)|u|^2dx\geqq 0,\end{aligned}$$

which is a contradiction.

3. Proof of Lemma 2. Since for $\lambda>0$, this lemma is proved in T. Kato [3], we may investigate this lemma when $\lambda\leqq 0$. We replace the equation $-\Delta u+qu=\lambda u$ by polar coordinate (r, θ), then this equation becomes

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{Au}{r^2}+(\lambda-q)u=0,$$

where A is Laplace-Beltrami operator. We now define $w(r,\theta)=ru(r,\theta)$, then $w(r,\theta)$ satisfies

$$\frac{\partial^2 w}{\partial r^2}+\frac{Aw}{r^2}+(\lambda-q)w=0.$$

By $\varphi_{n,m}(\theta)$ we denote normalized n-th spherical harmonics, then $w(r,\theta)$ is expanded such that

$$w(r,\theta)=\sum_{n=0}^{\infty}\sum_{m}\int_{|\theta|=1} w(r,\theta)\overline{\varphi_{n,m}(\theta)}d\theta\cdot\varphi_{n,m}(\theta),$$

and its coefficient $v_{n,m}(r)=\int_{|\theta|=1} w(r,\theta)\overline{\varphi_{n,m}(\theta)}d\theta$ satisfies the equation

$$v''_{n,m}(r)+\Big(\lambda-q-\frac{n(n+1)}{r^2}\Big)v_{n,m}(r)=0.$$

We at first show that $v_{n,m}(r)\equiv 0$ for $n\geqq 1$. If $v_{n,m}\not\equiv 0$, from $u(x)\in L^2(R^3)$, $\frac{\partial u}{\partial x_i}\in L^2(R^3)$, $(i=1,2,3)$

$$\begin{aligned}0&=-\int_0^\infty\Big\{v''_{n,m}+\Big(\lambda-q-\frac{n(n+1)}{r^2}\Big)v_{n,m}\Big\}\bar v_{n,m}dr\\&=\int_0^\infty\Big\{|v'_{n,m}|^2+\Big(q-\lambda+\frac{n(n+1)}{r^2}\Big)|v_{n,m}|^2\Big\}dr\\&>\int_0^\infty\Big\{(\tfrac14+2)\frac{1}{r^2}-(\lambda+q_-)\Big\}|v_{n,m}|^2dr\geqq 0.\end{aligned}$$

This is a contradiction.

Next we show that $v_0(r)\equiv 0$. From the preceding fact

$$v''_0=(\mu+q)v_0 \tag{1}$$

where we write $\lambda=-\mu$ $(\mu\geqq 0)$.

When $\mu=0$, from (1) we get

$$v_0(r)=v(0,r)=ar+b \qquad \text{for } r\geqq r_1.$$

From this and $v(0,r)\in L^2(R^1)$, we see that

$$v(0,r)=0 \qquad \text{for } r\geqq r_1,$$

and accordingly that $v(0,r)\equiv 0$.

Now denoting the solution v_0 of the equation $v''_0=(q+\mu)v_0$ by $v(\mu,r)$, we set

$$\left.\begin{aligned}v(\mu,r)&=\rho(\mu,r)\sin\theta(\mu,r),\\v'(\mu,r)&=\rho(\mu,r)\cos\theta(\mu,r),\\\rho(\mu,r)&=\{v(\mu,r)^2+v'(\mu,r)^2\}^{\frac12},\end{aligned}\right\} \tag{2}$$

and

$$\theta(\mu,0)=0. \tag{3}$$

From (1) and (2), we get

$$\rho'(\mu,r)=(1+(\mu+q))\rho(\mu,r)\sin\theta(\mu,r)\cos\theta(\mu,r), \tag{4}$$

$$\theta'(\mu,r)=\cos^2\theta(\mu,r)-(\mu+q)\sin^2\theta(\mu,r) \tag{5}$$

for all $r\geqq 0$.

Solving (1) for $r\geqq r_1$, we get

$$v(\mu,r)=ae^{\sqrt{\mu}r}+be^{-\sqrt{\mu}r}. \tag{6}$$

Because of $u\in L^2(R^3)$, $a=0$. Therefore $v(\mu, r)=be^{-\sqrt{\mu}r}$ for $r\geqq r_1$. Assuming $b\neq 0$ for some $\mu>0$, from (2), (4), (6), we get

$$\sin 2\theta(\mu, r)=-\frac{2\sqrt{\mu}}{1+\mu} \qquad \text{for } r\geqq r_1.$$

Accordingly for some integer k

$$\theta(\mu, r)\in((k+\tfrac{1}{2})\pi, (k+1)\pi), \tag{7}$$

and for $r\geqq r_1$, $\theta(\mu, r)$ is constant. Hence from (5)

$$\mu=\cot^2(\mu, r), \qquad \text{for } r\geqq r_1. \tag{8}$$

Here we remark from (5), (8) that even if there exists an eigenfunction whose $\theta(\mu, r)$ is in $((k+\frac{1}{2})\pi, (k+1)\pi)$ for $r\geqq r_1$, it is determined by a unique μ.

Now we assume that there exists a positive solution $v(0, r)$. Setting $v(0, r)=ar+b$ for $r\geqq r_1$, we see that $\frac{1}{2}\sin 2\theta(0, r)=\dfrac{\rho'}{\rho}\to 0$ as $r\to\infty$, hence that $\theta(0, r)\to\dfrac{k\pi}{2}$ as $r\to\infty$. Furthermore it implies from the positiveness of $v(0, r)$ that $\sin\theta(0, r)\neq 0$, that is, $0\lneqq\theta(0, r)\lneqq\pi$. Moreover from (5), we see that

$$0\lneqq\theta(0, r)\leqq\frac{\pi}{2} \qquad \text{for } r\in(r_1, \infty). \tag{9}$$

If $\mu>0$, from (3), (5), and (9)

$$0\leqq\theta(\mu, r)\lneqq\theta(0, r) \qquad \text{for all } r>0.$$

that is, for $r>0$, $\mu>0$, $\theta(\mu, r)\in\left[0, \dfrac{\pi}{2}\right]$, which is a contradiction with (7).

Finally we remark that if $\theta(0, r)$ tends to $(k+\frac{1}{2})\pi$, as $r\to\infty$, then from (8) the operator L has just k-eigenvalues with simple multiplicity.

§ 3. **Remark.** For the dimension $n=3$, there exists at least one operator L which has eigenvalues even if q satisfies $q_-\leqq(\frac{1}{4}+\varepsilon)\dfrac{1}{r^2}$, where ε is an arbitrary positive number.

Example 2. We set

$$q(r)=\begin{cases}-k^2-\lambda & \text{for } 0\leqq r<r_1,\\ -(\frac{1}{4}+\delta)\dfrac{1}{r^2}-\lambda & \text{for } r_1\leqq r\leqq r_2,\\ 0 & \text{for } r>r_2,\end{cases}$$

and

$$w(r)=\begin{cases}\sin kr & \text{for } 0\leqq r<r_1,\\ r^{\frac{1}{2}}\sin(\sqrt{\delta}\log r) & \text{for } r_1\leqq r\leqq r_2,\\ e^{-\sqrt{\lambda}r} & \text{for } r>r_2,\end{cases}$$

where δ is a fixed number such that $0<\delta\leqq\dfrac{\varepsilon}{2}$, and we choose k, r_1

such that

$$k>0,$$
$$r_1>0,$$
$$\sin kr_1 = r_1^{\frac{1}{2}} \sin(\sqrt{\delta}\log r_1),$$
$$k\cot kr_1 = \frac{1}{r_1}\left\{\sqrt{\delta}(\cot\sqrt{\delta}\log r_1)+\tfrac{1}{2}\right\},$$

in addition k^2 is sufficiently smaller than $(\frac{1}{4}+\varepsilon)\frac{1}{r_1^2}$. Next we choose λ, r_2 such that

$$r_1<r_2,$$
$$0<\lambda\leqq\frac{\varepsilon}{2r_2^2},$$
$$r_2^{\frac{1}{2}}\sin(\sqrt{\delta}\log r_2)=e^{-\sqrt{\lambda}r_2},$$

and
$$\frac{1}{r_2}\left\{\sqrt{\delta}\cot(\sqrt{\delta}\log r_2)+\tfrac{1}{2}\right\}=-\sqrt{\lambda}.$$

Then $q(r)$ and $w(r)$ mentioned above, satisfy the relation

$$w''(r)=(q+\lambda)w(r).$$

We next set $u(x)=r^{-1}w(r)$, then

$$Lu=\lambda u, \quad \text{and also} \quad u\in L^2(R^3).$$

References

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