220. A Note on the Inductive Dimension of Product Spaces

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The *large inductive dimension* of a space[1] X, Ind X, is defined inductively as follows. If X is the empty set, Ind $X=-1$. For $n=0,1,2,\cdots$, Ind $X\leqq n$ means that for any pair of a closed set F and an open set G with $F\subset G$ there exists an open set U such that $F\subset U\subset G$, Ind $(\bar{U}-U)\leqq n-1$. Ind $X=n$ means that Ind $X\leqq n$ and the statement Ind $X\leqq n-1$ is false. Ind $X=\infty$ means that the statement Ind $X\leqq n$ is false for any n.

In [11], K. Nagami proved that the inequality

$$\text{(1)} \qquad \operatorname{Ind}(X\times Y)\leqq \operatorname{Ind} X+\operatorname{Ind} Y$$

holds for the case where X is a perfectly normal paracompact space and Y is a metric space. Then N. Kimura [5] generalized the above result of Nagami by proving that the inequality (1) holds for the case where Y is a metric space and $X\times Y$ is a countably paracompact, totally normal space. Here the notion of totally normal spaces was defined by C. H. Dowker [3] and he proved that the subset theorem and the sum theorem hold for the large inductive dimension of totally normal spaces.

On the other hand, as for the covering dimension of product spaces, K. Morita [9] proved that the inequality

$$\dim(X\times Y)\leqq \dim X+\dim Y$$

holds for the following three cases: (a) $X\times Y$ is an S-space, where a space R is said to be an *S-space* if every open covering of R has a star-finite open refinement, (b) X is a paracompact space and Y is a locally compact paracompact space and (c) X is a countably paracompact normal space and Y is a locally compact metric space.

In this note we shall prove that the above inequality (1) holds for the following two cases:

I. $X\times Y$ is a totally normal S-space.

II. X is a paracompact space, Y is a locally compact paracompact space and $X\times Y$ is a totally normal space.

Our proof for Case I is based on the fact that if R is a totally normal S-space we have Ind $R=$ ind R. Here the *small inductive dimenion* of a space X, ind X, is defined inductively as follows. If X is the empty set, ind $X=-1$. For $n=0,1,2,\cdots$, ind $X\leqq n$

1) Throughout this note a *space* means a Hausdorff space.

means that for any pair of a point x and an open set G with $x \in G$ there exists an open set U such that $x \in U \subset G$, ind $(\bar{U}-U) \leqq n-1$. ind $X=n$ means that ind $X \leqq n$ and the statement ind $X \leqq n-1$ is false. ind $X=\infty$ means that the statement ind $X \leqq n$ is false for any n.

We can prove the following lemma by the argument as in the proof of [11, Lemma 2].

Lemma. *In a totally normal paracompact space X the following conditions are equivalent.*

i) Ind $X \leqq n$.

ii) *Every open covering can be refined by a locally finite open covering $\mathfrak{V}$ such that for any $V \in \mathfrak{V}$ Ind $(\bar{V}-V) \leqq n-1$.*

iii) *Every open covering can be refined by a σ-locally finite*[2] *open covering $\mathfrak{V}$ such that for any $V \in \mathfrak{V}$ Ind $(\bar{V}-V) \leqq n-1$.*

Theorem 1. *If X is a totally normal S-space, then*

$$\text{Ind } X = \text{ind } X.$$

Proof. Since the inequality Ind $X \geqq$ ind X is obvious, we shall prove only the inequality

$$\text{Ind } X \leqq \text{ind } X. \tag{2}$$

When ind X is infinite, (2) is evidently true. Hence we prove that (2) holds for the case ind $X<\infty$. We shall carry out the proof by induction on $n=$ ind X. When $n=-1$, X is empty and hence Ind $X=$ ind X. Let us assume that (2) is true if ind $X \leqq n-1$. We want to show that (2) is true in the case ind $X=n$.

Let $\mathfrak{G}$ be an arbitrary open covering of X. Let us construct a refinement of $\mathfrak{G}$ satisfying the condition iii) of Lemma. For each point x of X there exists an open neighbourhood $U(x)$ such that $U(x)$ is contained in some open set of $\mathfrak{G}$ and ind $(\overline{U(x)}-U(x)) \leqq n-1$. Since X is an S-space, the open covering $\{U(x); x \in X\}$ has a star-finite open refinement $\mathfrak{V}$. Because of the star-finiteness we may assume that the open covering $\mathfrak{V}$ consists of $V_{\alpha i}$, $\alpha \in \Omega$, $i=1,2,\cdots$, such that

$$V_{\alpha i} \cap V_{\beta j}=\phi, \text{ for } \alpha \neq \beta.$$

If we put

$$V_\alpha=\bigcup_{i=1}^{\infty} V_{\alpha i}$$

then V_α is at the same time closed and open. By the construction, each $V_{\alpha i}$ is contained in the neighbourhood $U(x_{\alpha i})$ of some point $x_{\alpha i}$ of X. If we put

$$W_{\alpha i}=V_\alpha \cap U(x_{\alpha i}),$$

then it is obvious that

2) A collection of subsets of a space is said to be *σ-locally finite* if it is the union of a countable number of locally finite subcollections.

$$\bar{W}_{\alpha i}-W_{\alpha i}\subset\overline{U(x_{\alpha i})}-U(x_{\alpha i}).$$

Therefore we have

$$\operatorname{ind}(\bar{W}_{\alpha i}-W_{\alpha i})\leqq n-1.$$

Since each $\bar{W}_{\alpha i}-W_{\alpha i}$ is a closed set of X, it is also a totally normal S-space. By the assumption of the induction, we have

$$\operatorname{Ind}(\bar{W}_{\alpha i}-W_{\alpha i})\leqq n-1.$$

It is easily proved that the collection $\{W_{\alpha i};\alpha\in\Omega, i=1,2,\cdots\}$ is a σ-locally finite open refinement of $\mathfrak{G}$. Thus, by Lemma we have Ind $X\leqq n$, and the proof is completed.

Theorem 2. *Let $X\times Y$ be a totally normal S-space. If at least one of X and Y is not empty, then we have*

$$\operatorname{Ind}(X\times Y)\leqq\operatorname{Ind}X+\operatorname{Ind}Y.$$

Proof. By Theorem 1 we have

$$\operatorname{Ind}(X\times Y)=\operatorname{ind}(X\times Y).$$

As is well known, for arbitrary spaces X and Y we have

$$\operatorname{ind}(X\times Y)\leqq\operatorname{ind}X+\operatorname{ind}Y.$$

And the inequalities

$$\operatorname{ind}X\leqq\operatorname{Ind}X,\ \operatorname{ind}Y\leqq\operatorname{Ind}Y$$

are evidently true. Thus we obtain the theorem.

Theorem 3. *Let X be a paracompact space, Y a locally compact paracompact space and $X\times Y$ a totally normal space. If at least one of X and Y is not empty, then we have*

$$\operatorname{Ind}(X\times Y)\leqq\operatorname{Ind}X+\operatorname{Ind}Y.$$

Proof. It is known that $X\times Y$ is paracompact ([9, Theorem 4]).

First we consider the case where Y is compact. When Ind X or Ind Y is infinite, the theorem is evidently true. Hence we prove the theorem for the case Ind $X<\infty$, Ind $Y<\infty$. We shall carry out the proof by induction on $k=\operatorname{Ind}X+\operatorname{Ind}Y$. When $k=-1$, either X or Y is empty and hence the theorem is true. Let us assume that the theorem is true if $\operatorname{Ind}X+\operatorname{Ind}Y<k$. Let Ind $X=m$, Ind $Y=n$, and $m+n=k$.

Let $\mathfrak{G}$ be an arbitrary open covering of $X\times Y$. Let us construct a refinement of $\mathfrak{G}$ satisfying the condition ii) of Lemma. Let x be any fixed point of X. Each point (x,y) of $x\times Y$ is contained in a set $U\times V$ such that U is an open set of X, V is an open set of Y, $U\times V$ is contained in some open set of $\mathfrak{G}$, and $\operatorname{Ind}(\bar{V}-V)\leqq n-1$. The collection of all such V's is an open covering of the compact space Y, and hence Y is covered by a finite number of them, $V_{x,1}, V_{x,2},\cdots, V_{x,q(x)}$. Let $U(x)$ be the intersection of the correspoding U's, then $U(x)$ is an open set of X. By Lemma the open covering $\{U(x); x\in X\}$ of X has a locally finite open refinement $\{U_\alpha;\alpha\in\Omega\}$ such that for any α $\operatorname{Ind}(\bar{U}_\alpha-U_\alpha)\leqq m-1$. Correspoding to each U_α we choose x such that $U_\alpha\subset U(x)$ and we put

$$V_{\alpha i}=V_{x,i}, \qquad i=1,2,\cdots,q_\alpha,$$

where $q_\alpha=q(x)$. Then the collection $\{U_\alpha\times V_{\alpha i};\ \alpha\in\Omega,\ i=1,2,\cdots,q_\alpha\}$ is a locally finite open refinement of $\mathfrak{G}$. Since $\bar U_\alpha$ and $\bar V_{\alpha i}$ are closed sets of X and Y respectively, we have

$$\operatorname{Ind}\bar U_\alpha\leqq m,\ \operatorname{Ind}\bar V_{\alpha i}\leqq n.$$

On the other hand, by the construction we have

$$\operatorname{Ind}(\bar U_\alpha-U_\alpha)\leqq m-1,\ \operatorname{Ind}(\bar V_{\alpha i}-V_{\alpha i})\leqq n-1.$$

Therefore, by the assumption of the induction, we obtain

$$\operatorname{Ind}((\bar U_\alpha-U_\alpha)\times\bar V_{\alpha i}),\ \operatorname{Ind}(\bar U_\alpha\times(\bar V_{\alpha i}-V_{\alpha i}))\leqq m+n-1=k-1.$$

Since $\overline{U_\alpha\times V_{\alpha i}}-U_\alpha\times V_{\alpha i}=((\bar U_\alpha-U_\alpha)\times\bar V_{\alpha i})\cup(\bar U_\alpha\times(\bar V_{\alpha i}-V_{\alpha i}))$, applying the sum theorem for the large inductive dimension ([3, Theorem 3]) we have

$$\operatorname{Ind}(\overline{U_\alpha\times V_{\alpha i}}-U_\alpha\times V_{\alpha i})\leqq k-1.$$

Hence the requiered inequality $\operatorname{Ind}(X\times Y)\leqq k$ is obtained by Lemma. Consequently the theorem is established by induction for the case where Y is compact.

Next, let Y be a locally compact paracompact space. Then each point y of Y has a neighbourhood $V(y)$ whose closure $\overline{V(y)}$ is compact. As is shown above, we have

$$\operatorname{Ind}(X\times\overline{V(y)})\leqq\operatorname{Ind}X+\operatorname{Ind}\overline{V(y)}\leqq\operatorname{Ind}X+\operatorname{Ind}Y$$

for any $y\in Y$. Therefor by [4, Proposition [3.4]] we have Ind $(X\times Y)\leqq\operatorname{Ind}X+\operatorname{Ind}Y$. Thus the proof is completed.

Remark. The product space of two totally normal spaces is not totally normal (even normal) in general. Indeed, as the example given by R. H. Sorgenfrey [12] or E. Michael [6] shows, the product space $X\times Y$ is not necessarily normal, even if both X and Y are perfectly normal S-spaces with the Lindelöf property, or even if X is a hereditarily paracompact space with the Lindelöf property and Y is a separable metric space. (Notice that a perfectly normal space as well as a hereditarily paracompact space is totally normal [3]).

It is known that $X\times Y$ is perfectly normal (and hence totally normal) in the following cases:

(A) X is a perfectly normal space and Y is a metric space (Morita [10]).

(B) Both X and Y are M_3-spaces in the sense of J. G. Ceder [2], or equivalently, stratifiable spaces in the sense of C.J.R. Borges [1].

(C) Both X and Y are cosmic[3] spaces in the sense of Michael

3) According to Michael [7], a collection $\mathfrak{P}$ of (not necessarily open) subsets of a space X is said to be a *point-pseudobase* for X if any pair of a point x and an open set U with $x\in U$ there exists a set P of $\mathfrak{P}$ such that $x\in P\subset U$, and a space is said to be *cosmic* if it is a regular space with a countable point-pseudobase.

[7].

Kimura [5] proved the inequality (1) mentioned above holds in Case (A).

Next, we see that $X \times Y$ is an S-space in Case (C). Indeed, the product space of two cosmic spaces is cosmic, a cosmic space has the Lindelöf property [7], and a regular space with the Lindelöf property is an S-space [8]. Similarly, $X \times Y$ is an S-space in the following case:

(D) Both X and Y are separable M_3-spaces.

Therefore the inequality (1) holds in Cases (C) and (D) by our Theorem 2.

Finally, we can prove that $X \times Y$ is also perfectly normal in the following case:

(E) X is a cosmic space (more generally, a paracompact space with a σ-locally finite[4)] point-pseudobase[5)]) and Y is a locally compact perfectly normal paracompact space.

Therefore the inequality (1) holds in Case (E) by our Theorem 3.

References

[1] C. J. R. Borges: On stratifiable spaces. Pacific J. Math., **17**, 1-16 (1966).
[2] J. G. Ceder: Some generalizations of metric spaces. Pacific J. Math., **11**, 105-125 (1961).
[3] C. H. Dowker: Inductive dimension of completely normal spaces. Quart. J. Math., **4**, 267-281 (1953).
[4] ——: Local dimension of normal spaces. Quart. J. Math., **6**, 101-120 (1955).
[5] N. Kimura: On the inductive dimension of product spaces. Proc. Japan Acad., **39**, 641-646 (1963).
[6] E. Michael: The product of a normal space and a metric space need not be normal. Bull. Amer. Math. Soc., **69**, 375-376 (1963).
[7] ——: $\aleph_0$-spaces (to appear).
[8] K. Morita: Star-finite coverings and the star-finite property. Math. Japonicae, **1**, 60-68 (1948).
[9] ——: On the dimension of product spaces. Amer. J. Math., **75**, 205-223 (1953).
[10] ——: On the product of a normal space with a metric space. Proc. Japan Acad., **39**, 148-150 (1963).
[11] K. Nagami: On the dimension of product spaces. Proc. Japan Acad., **36**, 560-564 (1960).
[12] R. H. Sorgenfrey: On the topological product of paracompact spaces. Bull. Amer. Math. Soc., **53**, 631-632 (1947).

4) See Footnote 2).
5) See Footnote 3).