246. A Note on Multipliers of Ideals in Function Algebras

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Let X be a compact Hausdorff space and let $C(X)$ be the algebra of all complex-valued continuous functions on X. By a function algebra we mean a closed (by supremum norm) subalgebra in $C(X)$ containing constants and separating points of X. Recently J. Wells [7] has obtained interesting theorems on multipliers of ideals in function algebras. And especially in the disc algebra A_1 it was shown that for any non-zero closed ideal J in A, $\mathfrak{M},(J)$ is the set of all H^∞-functions continuous on $D\sim F$, where D is the closed unit disc on the complex plane and F is the intersection of the zeros of the functions in J on the unit circle C ([7], Theorem 8). As A_1 is an essential maximal algebra, the question naturally arises: Does a similar theorem hold for arbitrary essential maximal algebra? The main purpose of this note is to answer the question under certain conditions and to give a generalization of the theorem mentioned above (cf. Theorem 2).

1. Let A be a function algebra on a compact Hausdorff space X. Let J be a non-zero closed ideal in A. By a *multiplier* of J we mean a function φ on $X\sim h(J)$ such that $\varphi J\subset J$, where $h(J)$, the hull of J, is the set of points at which every function in J vanishes. Every multiplier of J is a bounded continuous function on the locally compact space $X\sim h(J)$. We denote the set of all multipliers of J by $\underline{\mathfrak{M}}(J)$. $M(X)$ denotes the set of all complex, finite, regular Borel measures μ on X and a $\mu(\in M(X))$ is orthogonal to A ($\mu\perp A$) means $\int f d\mu=0$ for any $f\in A$. For μ in $M(X)$, μ_F denotes the restriction of μ to F. $C(Y)_\beta$ denotes the space of bounded continuous functions on the locally compact space Y under the strict topology β of Buck ([3], [7]). Let A be a function algebra on X and let F be a closed subset of X. Then F is said to have the *condition* (P) if $\mu_F\perp A$ for every $\mu\perp A$. If F has (P), it is an intersection of peak sets ([4]). Wells [7] has proved the following theorem: $\underline{\mathfrak{M}}(kF)$ is the closure of kF in $C(X\sim F)_\beta$ if and only if F has (P), where $kF=\{f\in A: f(F)=0\}$. Let $F_0=h(J)$, then $\underline{\mathfrak{M}}(kF_0, J)$ denotes the set of all functions φ on $X\sim F_0$ such that $\varphi\cdot kF_0\subset J$. Every function in $\underline{\mathfrak{M}}(kF_0, J)$ is a bounded continuous

function on $X\sim F_0$.

Theorem 1. *Let J be a non-zero closed ideal in a function algebra on X and let $F_0=h(J)$. If F_0 has (P), then $\mathfrak{M}(kF_0, J)$ is the closure of J in $C(X\sim F_0)_\beta$.*

Proof. Since F_0 has (P), F_0 is an intersection of peak sets, so we can set $F_0=\bigcap_j F_j$, $F_j=\{x: g_j(x)=1\}$, where every g_j is a function in A such that $|g_j(x)|<1$ if $x\notin F_j$. If we put

$$h_{j_1,j_2,\cdots,j_m,n}=1-(g_{j_1}g_{j_2}\cdots g_{j_m})^n ,$$

then these functions are contained in $k(F_0)$. We can here prove that $h_{j_1,j_2,\cdots,j_m,n}$ converges to 1 under the topology β, where the ordering $>$ of the directed set $\{(j_1, j_2, \cdots, j_m, n)\}$ is the following; $(j_{p_1}, j_{p_2}, \cdots, j_{p_s}, n)>(j_{q_1}, j_{q_2}, \cdots, j_{q_t}, n')$ if the finite set $j_{p_1}, j_{p_2}, \cdots, j_{p_s} \supset j_{q_1}, j_{q_2}, \cdots, j_{q_t}$ and $n\geqq n'$. For, let f be a continuous function on $X\sim F_0$ which vanishes at infinity. Then if we put $f(x)=0$ for any $x\in F_0$, f can be regarded a continuous function on X. We set $U_0=\{x\in X: |f(x)|<\varepsilon\}$. Since $\bigcap_j F_j=F_0\subset U_0$, $F_{j_1}\cap F_{j_2}\cap\cdots\cap F_{j_m}\subset U_0$ for some $j_1, j_2, \cdots, j_m$ and

$$\sup\{|(g_{j_1}g_{j_2}\cdots g_{j_m})(x)|;\ x\in X\sim U_0\}<1.$$

Therefore $\|(g_{j_1}g_{j_2}\cdots g_{j_m})^n f\|_\infty<\varepsilon$ for any $n>$ some n_0. This shows that $\|(1-h_{j_1,j_2,\cdots,j_m,n})f\|_\infty\to 0$. Now let φ be any function in $\mathfrak{M}(kF_0, J)$, then $\varphi h_{j_1,j_2,\cdots,j_m,n}\in J$ and $\varphi h_{j_1,j_2,\cdots,j_m,n}$ converges to φ under the topology β, so $\mathfrak{M}(kF_0, J)$ is contained in the closure $\bar{J}^\beta$ of J under β. Conversely, it is obvious that $\mathfrak{M}(kF_0, J)\supset\bar{J}^\beta$.

Remark. As we see in the proof of Theorem 1, for any $\varphi\in\mathfrak{M}(kF_0, J)$ $\varphi h_{j_1,j_2,\cdots,j_m,n}$ converges to φ under β. We see here that $\|\varphi h_{j_1,j_2,\cdots,j_m,n}\|_\infty\leqq\|\varphi\|_\infty\|h_{j_1,j_2,\cdots,j_m,n}\|_\infty\leqq 2\|\varphi\|_\infty$.

2. Let A be a function algebra on X and let $S(A)$ be the maximal ideal space of A. Let J be a non-zero closed ideal in A. Since A can be regarded as a function algebra on $S(A)$, we denote the function algebra by $\hat{A}$: $\hat{A}=\{\hat{f}: f\in A\}$ and $\hat{f}(m)=m(f)$ for any $m\in S(A)$, in other words, for any non-zero homomorphism m on A. $\hat{J}=\{\hat{f}: f\in J\}$ is a closed ideal in $\hat{A}$ and $\mathfrak{M}(\hat{J})$ can be defined as a subalgebra of $C(S(A)\sim\hat{F}_0)$, where $\hat{F}_0=h(\hat{J})$. We shall use the symbol $\mathfrak{M}(J)$ in the place of $\mathfrak{M}(\hat{J})$. $H^\infty_{F_0}$ is the set of all bounded continuous functions u on $S(A)\sim F_0$ having the following condition; there is a net $\{\hat{u}_\lambda\}$ in $\hat{A}$ which is uniformly bounded ($\|\hat{u}_\lambda\|_\infty\leqq$some M), and $\hat{u}_\lambda$ converges uniformly to u on every compact subset in $S(A)\sim F_0$.

Let A_1 be the disc algebra, that is, the algebra of all continuous functions on $C=\{z: |z|=1\}$ with continuous extensions to $D=\{z: |z|\leqq 1\}$, analytic in the interior of D. Wells [7] has proved the following theorem: Let J be a non-zero closed ideal in A_1 and let F be the intersection of zeros of the functions in J on C. Then $\mathfrak{M}(J)$ is the set of all H^∞ functions continuous on $D\sim F$. We see here that F has (P) since F has Lebesgue measure zero, and $h(\hat{J})$ is non dense in D (cf. [5]). Moreover, we easily see that in the disc algebra A_1 H_F^∞ is equal to the set of all H^∞ functions continuous on $D\sim F$.

Following theorem is a generalization of the theorem mentioned above.

Theorem 2. *Let A be an essential maximal algebra and let J be a non-zero closed ideal in A. If $F_0=h(J)$ has (P) (cf. § 1) and if $\hat{F}_0=h(\hat{J})$ is non dense in $S(A)$, then $\mathfrak{M}(J)=\mathfrak{M}(kF_0)=H_{F_0}^\infty$.*

Although any function φ is $\mathfrak{M}(J)$ is a continuous function on $S(A)\sim\hat{F}_0$, it can be extended continuously to a unique function in $C(S(A)\sim F_0)$. By a function φ in $\mathfrak{M}(J)$ in the above theorem, we mean the extended function of φ. Suppose that φ is a function in $\mathfrak{M}(J)$. To show that φ can be extended continuously to a function in $C(S(A)\sim F_0)$, we put $F_0=\bigcap_\alpha F_\alpha$, where $F_\alpha=\{x\colon g_\alpha(x)=1\}$, g_α is a function in A and $|g_\alpha(x)|<1$ if $x\notin F_\alpha$. If $h_\alpha=1-g_\alpha$, $h_\alpha(F_\alpha)=0$, and $h_\alpha(x)\neq 0$ for $x\in X\sim F_\alpha$. If ψ is the restriction of φ to X, $\psi\in\mathfrak{M}(J)$ and ψh_α is a continuous function on X. Since $\psi h_\alpha J\subset\psi J\subset J$, by Wells ([7], Theorem 7), $\psi h_\alpha\in A$. On the other hand, for any $\hat{f}$ in $\hat{J}$, $\varphi\hat{f}\hat{h}_\alpha=\hat{g}\in\hat{J}$. If we set $h_\alpha\psi=p_\alpha$, then $fp_\alpha=g$, $\hat{f}\hat{p}_\alpha=\hat{g}$, and $\varphi\hat{f}\hat{h}_\alpha=\hat{f}\hat{p}_\alpha$. Since $\hat{f}(\in\hat{A})$ is arbitrary, $\varphi\hat{h}_\alpha=\hat{p}_\alpha$ on $S(A)\sim\hat{F}_0$. Since $\hat{h}_\alpha$ never vanishes on $S(A)\sim F_\alpha$ (cf. [2]), $\rho_\alpha(x)=\hat{p}_\alpha(x)/\hat{h}_\alpha(x)$ is continuous on $S(A)\sim F_\alpha$. Since $S(A)\sim\hat{F}_0$ is dense in $S(A)\sim F_\alpha$ and φ is equal to ρ_α on $S(A)\sim(\hat{F}_0\cup F_\alpha)$ for any α, φ can be extended to a function in $C(S(A)\sim F_0)$.

We first prove the following lemmas.

Lemma 1. *If $F_0=h(J)$ has (P), then $\mathfrak{M}(J)\supset\mathfrak{M}(kF_0)$.*

Proof. If $\varphi\in\mathfrak{M}(kF_0)$, then $\varphi\cdot\widehat{kF}_0\subset\widehat{kF}_0$, where $\widehat{kF}_0=\{\hat{f}: f\in kF_0\}$. For any $\hat{f}\in\hat{J}$, we have $\varphi\hat{f}\cdot\widehat{kF}_0\subset\hat{f}\cdot\widehat{kF}_0\subset\hat{J}$. Since F_0 has (P), there is a net $\{u_j\}$ in kF_0 such that u_j converges to 1 under the topology β, so u_jf converges to f uniformly. Since $\hat{u}_j\hat{f}$ converges to $\hat{f}$ uniformly and $\hat{u}_j\in\widehat{kF}_0$, $\varphi\hat{f}\in\hat{J}$ and $\varphi\hat{J}\subset\hat{J}$.

Lemma 2. *If A is an essential maximal algebra and if $\hat{F}_0=h(\hat{J})$ is non dense in $S(A)$, $\mathfrak{M}(J)\subset\mathfrak{M}(kF_0)$.*

Proof. If $\varphi\in\mathfrak{M}(J)$, $\varphi\hat{J}\subset\hat{J}$. For any $\hat{\alpha}\in\widehat{kF}_0$, $\varphi\hat{\alpha}$ is continuous on $S(A)$ and $\varphi\hat{\alpha}\hat{J}\subset\hat{\alpha}\hat{J}\subset\hat{J}$, so $\varphi\hat{\alpha}\in\mathfrak{M}(J)$. By the following lemma, $\varphi\hat{\alpha}\in\hat{A}$, and since $\varphi\hat{\alpha}(F_0)=0$, $\varphi\hat{\alpha}\in\widehat{kF}_0$. This shows that $\varphi\in\mathfrak{M}(kF_0)$.

We shall prove the following lemma, which is similar to a theorem of Wells ([7], Theorem 7).

Lemma 3. *If $\hat{F}_0=h(\hat{J})$ is non dense in $S(A)$ and if A is an essential maximal algebra, then any function φ in $\mathfrak{M}(J)$ which* *can be extended continuously to a function in* $C(S(A))$ *is in* $\hat{A}$.

Proof. We set $B=\{f\in C(S(A))$: the restriction of f to $S(A)\sim\hat{F}_0$ is in $\mathfrak{M}(J)\}$. Then we easily see that B is a closed subalgebra in $C(S(A))$ and $B\supset\hat{A}$. To prove that the Šilov boundary ∂_B of B is equal to $\partial_A(=X)$, it suffices to show that the Choquet boundary M_B of B is contained in ∂_A ([6], p. 40). If $x_0\in M_B$, then for any neighborhood $V(x_0)$ of x_0 in $S(A)$ there is a function $f_0\in B$ such that $|f_0(x_0)|>1$ and $|f_0(x)|\leqq$ some $\eta<1$ for any $x\in S(A)\sim V(x_0)$. Take a neighborhood $W(x_0)$ of x_0 in $S(A)$ such that $V(x_0)\supset W(x_0)$ and $|f_0(x)|>1$ for any $x\in W(x_0)$, then there is a point $x'\in W(x_0)\sim\hat{F}_0$, since $\hat{F}_0$ is non dense in $S(A)$. Since $\hat{F}_0\not\ni x'$ there is a function $\hat{g}\in\hat{J}$ such that $\hat{g}(x')\neq 0$, and we can here assume that $\hat{g}(x')=1$. If we set $\hat{h}=\hat{g}f_0^n$, then $\hat{h}\in\hat{J}$ and for a sufficiently large n, $|\hat{h}(x')|>1$ and $|\hat{h}(y)|\leqq$ some $\eta'<1$ for any y in $S(A)\sim V(x_0)$, so $x_0\in\partial_A$. Now if B_1 is the restriction of B to X, $A\subset B_1\subset C(X)$. Since A is maximal, it follows that $A=B_1$ or $B_1=C(X)$. If $A=B_1$, we obviously see that $B=\hat{A}$ since $\partial_A=\partial_B=X$. In this case, any $f\in C(S(A))$ whose restriction to $S(A)\sim\hat{F}_0$ is in $\mathfrak{M}(J)$ is a function of $\hat{A}$. Next we shall show that B_1 is not equal to $C(X)$. Assume the contrary and let $\hat{f}_0$ be a non-zero fixed function in $\hat{J}$, then $Z(\hat{f}_0)\not\supset X$, where $Z(\hat{f}_0)=\{x\in S(A), \hat{f}_0(x)=0\}$. We take an open set U in X such that $Z(\hat{f}_0)\cap X\subset U\subset\bar{U}\subsetneqq X$, where $\bar{U}$ is the closure of U in X. We shall prove here that any function $f\in C(X)$ such that $f(\bar{U})=0$ is in A. If this were proved, A would be not an essential algebra ([1]). This contradiction shows that B_1 is not equal to $C(X)$. Let f be a function in $C(X)$ such that $f(\bar{U})=0$. Since f_0 never vanishes on $X\sim U$, f_0^{-1} can be extended continuously to a function h of $C(X)$. If we put $\varphi=fh$, then $\varphi\in C(X)$ and $\varphi f_0=f$. Since $B_1=C(X)$, $\varphi=b$ on X for some $b\in B$, and $b\hat{f}_0\in b\hat{J}\subset\hat{J}\subset\hat{A}$ and $f=f_0\varphi\in A$.

The proof of Theorem 2. By Lemmas 1 and 2, it remains only to prove that $\mathfrak{M}(kF_0)=H^\infty_{F_0}$. Since F_0 has (P), F_0 is an intersection of peak sets in X. Since A is an essential maximal algebra, by Bear [2], F_0 is an intersection of peak sets in $S(A)$. By the remark of Theorem 1, for any u in $\mathfrak{M}(kF_0)$, there is a net $\{\hat{u}_k\}\subset\widehat{kF_0}$ such that $\|\hat{u}_k\|\leqq 2\|u\|$ and for any $\hat{f}\in\widehat{kF_0}$ $\hat{u}_k\hat{f}$ converges uniformly to $u\hat{f}$ on $S(A)$, so it is clear that $\mathfrak{M}(kF_0)\subset H^\infty_{F_0}$. Conversely, let u be a function in $H^\infty_{F_0}$, then there is a net $\{\hat{u}_j\}\subset\hat{A}$ such that $\|\hat{u}_j\|\leqq$ some M and $\hat{u}_j$ converges uniformly to u on every compact subset in $S(A)\sim F_0$. Since there is a net $\{\hat{\varphi}_k\}\subset\widehat{kF_0}$ such that $\|\hat{\varphi}_k\|\leqq 2$ and $\hat{\varphi}_k$ converges to 1 under β, we obviously see that the net $\{\hat{u}_j\hat{\varphi}_k\}\subset\widehat{kF_0}$ converges to u under the topology β on $S(A)\sim F_0$. This shows that

$H^{\infty}_{F_0} \subset \mathfrak{M}(kF_0)$.

Remark. In Theorem 2, if we assume "$\hat{A}$ is an analytic algebra" in the place of "$\hat{F}_0$ is non dense", the conclusion still holds.

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