241. *Some Theorems on Manifolds of Constant Curvature*

By Koichi OGIUE

Department of Mathematics, Tokyo Institute of Technology, Tokyo

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§1. Riemannian manifolds of constant curvature.

Let M be a connected Riemannian manifold with metric tensor g. We always assume that the dimension n of M is $\geqq 3$. Let ∇ be the covariant differentiation with respect to the Riemannian connection associated with g. The curvature tensor field R is given by

$$R(X, Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z,$$

where X, Y, and Z are vector fields on M.

Then we have

$$(1)\qquad R(X, Y)+R(Y, X)=0,$$

$$(2)\qquad R(X, Y)Z+R(Y, Z)X+R(Z, X)Y=0 \quad \text{(Bianchi's 1st identity)},$$

$$(3)\qquad (\nabla_X R)(Y, Z)+(\nabla_Y R)(Z, X)+(\nabla_Z R)(X, Y)=0 \quad \text{(Bianchi's 2nd identity)}.$$

The *Riemannian curvature tensor field* of M, denoted also by R, is the tensor field of covariant degree 4 defined by

$$R(X_1, X_2, X_3, X_4)=g(R(X_3, X_4)X_2, X_1).$$

Then R possesses the following properties:

$$(4)\qquad R(X_1, X_2, X_3, X_4)+R(X_2, X_1, X_3, X_4)=0,$$

$$(1')\qquad R(X_1, X_2, X_3, X_4)+R(X_1, X_2, X_4, X_3)=0,$$

$$(5)\qquad R(X_1, X_2, X_3, X_4)=R(X_3, X_4, X_1, X_2),$$

$$(2')\qquad R(X_1, X_2, X_3, X_4)+R(X_1, X_3, X_4, X_2)+R(X_1, X_4, X_2, X_3)=0,$$

$$(3')\qquad (\nabla_{X_5}R)(X_1, X_2, X_3, X_4)+(\nabla_{X_3}R)(X_1, X_2, X_4, X_5)+(\nabla_{X_4}R)(X_1, X_2, X_5, X_3)=0.$$

M is a *Riemannian manifold of constant curvature* if and only if

$$(6)\qquad R(X, Y)Z=k\{g(Y, Z)X-g(X, Z)Y\}$$

where k is a constant.

If R^i_{jkl} and g_{ij} are the components of the curvature tensor field and the metric tensor with respect to a local coordinate system, then the components R_{ijkl} of the Riemannian curvature tensor are given by

$$R_{ijkl}=\sum_{m=1}^{n} g_{im}R^m_{jkl}.$$

If M is a Riemannian manifold of constant curvature, then

$$R^i_{jkl}=k(\delta^i_k g_{jl}-\delta^i_l g_{jk})$$

or

$$R_{ijkl}=k(g_{ik}g_{jl}-g_{il}g_{jk}).$$

Theorem 1. *M is a Riemannian manifold of constant curvature if and only if $R(X, Y)Z$ is a linear combination of X and Y for every X, Y, and Z.*

Proof. If M is a Riemannian manifold of constant curvature, then the equation (6) means that $R(X, Y)Z$ is a linear combination of X and Y.

To prove the converse, let $R(X, Y)Z$ be a linear combination of X and Y for every X, Y, and Z. Then there exist two tensor fields α and β of covariant degree 2 such that

$$R(X, Y)Z=\alpha(Y, Z)X+\beta(X, Z)Y.$$

From (1) we have

$$\{\alpha(Y, Z)+\beta(Y, Z)\}X+\{\alpha(X, Z)+\beta(X, Z)\}Y=0.$$

Since X, Y, and Z are arbitrary, we get $\alpha+\beta=0$. Hence we have

$$R(X, Y)Z=\alpha(Y, Z)X-\alpha(X, Z)Y. \tag{7}$$

This, together with (2), implies

$$\alpha(X, Y)=\alpha(Y, X) \qquad \text{for every } X \text{ and } Y. \tag{8}$$

Let α_{ij} denote the components of α. Then (7) can be written as follows:

$$R^i_{jkl}=\alpha_{lj}\delta^i_k-\alpha_{kj}\delta^i_l \tag{7'}$$

or

$$R_{ijkl}=\alpha_{lj}g_{ik}-\alpha_{kj}g_{il}. \tag{7''}$$

This, together with (4), implies

$$\alpha_{lj}g_{ik}-\alpha_{kj}g_{il}+\alpha_{li}g_{jk}-\alpha_{ki}g_{jl}=0. \tag{9}$$

Let (g^{ij}) denote the inverse matrix of (g_{ij}). Multiplying (9) by g^{il} and summing with respect to i and l we obtain $\alpha=\frac{a}{n}g$, where $a=\sum_{i,l=1}^{n} g^{il}\alpha_{il}$. Hence we have

$$R(X, Y)Z=k\{g(Y, Z)X-g(X, Z)Y\},$$

that is,

$$R^i_{jkl}=k(\delta^i_k g_{jl}-\delta^i_l g_{jk}),$$

where $k=\frac{a}{n}$ is a function on M. This, together with (3), implies

$$k_{,m}(\delta^i_k g_{jl}-\delta^i_l g_{jk})+k_{,k}(\delta^i_l g_{jm}-\delta^i_m g_{jl})+k_{,l}(\delta^i_m g_{jk}-\delta^i_k g_{jm})=0,$$

where $k_{,m}$ denote the components of the covariant differential ∇k. Taking the trace with respect to i and m we obtain

$$k_{,l}g_{jk}-k_{,k}g_{jl}=0.$$

Multiplying by g^{jk} and summing with respect to j and k we have

$$k_{,l}=0.$$

Hence k is a constant.

Let M be a manifold with torsionfree affine connection and curvature tensor field R. It is natural to say that M is a *manifold of constant curvature* if $R(X, Y)Z$ is a linear combination of X and Y for every X, Y, and Z.

§ 2. Kählerian manifolds of constant holomorphic curvature.

Let M be a connected Kählerian manifold with complex structure J and with Kählerian metric g. We always assume that the real dimension $2n$ of M is $\geqq 4$. Let ∇ be the covariant differentiation with respect to the Kählerian connection associated with (J, g). Then the curvature tensor field R satisfies (1), (2), (3), and

$$R(JX, JY)=R(X, Y), \tag{10}$$

$$R(X, Y)JZ=JR(X, Y)Z. \tag{11}$$

M is a *Kählerian manifold of constant holomorphic curvature* if and only if

$$\begin{aligned} R(X, Y)Z=k\{&g(Y, Z)X-g(X, Z)Y \\ &+\Omega(Y, Z)JX-\Omega(X, Z)JY-2\Omega(X, Y)JZ\}, \end{aligned} \tag{12}$$

where Ω denotes the 2-form defined by $\Omega(X, Y)=g(JX, Y)$ for every X and Y and k is a constant.

Theorem 2. *M is a Kählerian manifold of constant holomorphic curvature if and only if $R(X, Y)Z$ is a linear combination of X, Y, JX, JY, and JZ for every X, Y, and Z.*

Proof. If M is a Kählerian manifold of constant holomorphic curvature, then the equation (12) means that $R(X, Y)Z$ is a linear combination of X, Y, JX, JY, and JZ.

To prove the converse, let $R(X, Y)Z$ be a linear combination of X, Y, JX, JY, and JZ for every X, Y, and Z. Then there exist five tensor fields α, β, λ, μ, and ν of covariant degree 2 such that

$$\begin{aligned} R(X, Y)Z=\alpha(Y, Z)X+\beta(X, Z)Y+\lambda(Y, Z)JX \\ +\mu(X, Z)JY+\nu(X, Y)JZ. \end{aligned}$$

From (10) we have

$$\begin{aligned} \{\alpha(Y, Z)+\lambda(JY, Z)\}X+\{\beta(X, Z)+\mu(JX, Z)\}Y \\ +\{\lambda(Y, Z)-\alpha(JY, Z)\}JX+\{\mu(X, Z)-\beta(JX, Z)\}JY \\ +\{\nu(X, Y)-\nu(JX, JY)\}JZ=0. \end{aligned}$$

Since X, Y, and Z are arbitrary, we get

$$\lambda(Y, Z)=\alpha(JY, Z), \tag{13}$$

$$\mu(X, Z)=\beta(JX, Z), \tag{14}$$

$$\nu(X, Y)=\nu(JX, JY) \tag{15}$$

for every X, Y, and Z.

From (1) we have

$$\begin{aligned} \{\alpha(Y, Z)+\beta(Y, Z)\}X+\{\alpha(X, Z)+\beta(X, Z)\}Y \\ +\{\lambda(Y, Z)+\mu(Y, Z)\}JX+\{\lambda(X, Z)+\mu(X, Z)\}JY \\ +\{\nu(X, Y)+\nu(Y, X)\}JZ=0. \end{aligned}$$

Since X, Y, and Z are arbitrary, we get

$$\alpha+\beta=0, \tag{16}$$

$$\lambda+\mu=0, \tag{17}$$

and

(18) $\nu(X, Y)+\nu(Y, X)=0$ for every X and Y.

Hence we have

$$R(X, Y)Z=\alpha(Y, Z)X-\alpha(X, Z)Y + \alpha(JY, Z)JX-\alpha(JX, Z)JY+\nu(X, Y)JZ.$$

This, together with (2), implies

(19) $$\alpha(X, Y)=\alpha(Y, X),$$

(20) $\nu(X, Y)=\alpha(X, JY)-\alpha(JX, Y)$ for every X and Y.

On the other hand, from (11) we have

(21) $\alpha(X, Y)=\alpha(JX, JY)$ for every X and Y.

This, together with (20), implies

(22) $$\nu(X, Y)=-2\alpha(JX, Y).$$

Hence we have

(23) $$R(X, Y)Z=\alpha(Y, Z)X-\alpha(X, Z)Y + \alpha(JY, Z)JX-\alpha(JX, Z)JY-2\alpha(JX, Y)JZ.$$

Let J_j^i, Ω_{ij}, and α_{ij} denote the components of J, Ω, and α respectively. Then (23) can be written as follows:

(23′) $$R^i_{jkl}=\alpha_{lj}\delta^i_k-\alpha_{kj}\delta^i_l+\sum_{a=1}^{2n}\alpha_{aj}J^a_lJ^i_k-\sum_{a=1}^{2n}\alpha_{aj}J^a_kJ^i_l-2\sum_{a=1}^{2n}\alpha_{al}J^a_kJ_{ij}$$

or

(23″) $$R_{ijkl}=\alpha_{lj}g_{ik}-\alpha_{kj}g_{il}+\sum_{a=1}^{2n}\alpha_{aj}J^a_l\Omega_{ik}-\sum_{a=1}^{2n}\alpha_{aj}J^a_k\Omega_{il}-2\sum_{a=1}^{2n}\alpha_{al}J^a_k\Omega_{ij}.$$

This, together with (4), implies

(24) $$\alpha_{lj}g_{ik}-\alpha_{kj}g_{il}+\sum_{a=1}^{2n}\alpha_{aj}J^a_l\Omega_{ik}-\sum_{a=1}^{2n}\alpha_{aj}J^a_k\Omega_{il} + \alpha_{li}g_{jk}-\alpha_{ki}g_{jl}+\sum_{a=1}^{2n}\alpha_{ai}J^a_l\Omega_{jk}-\sum_{a=1}^{2n}\alpha_{ai}J^a_k\Omega_{jl}=0.$$

Multiplying (24) by g^{il} and summing with respect to i and l and using (21) we obtain $\alpha=\frac{a}{2n}g$ where $a=\sum_{i,l=1}^{2n}g^{il}\alpha_{il}$. Hence we have

$$R(X, Y)Z=k\{g(Y, Z)X-g(X, Z)Y+\Omega(Y, Z)JX -\Omega(X, Z)JY-2\Omega(X, Y)JZ\},$$

where $k=\frac{a}{2n}$ is a function on M. By the similar way as in Theorem 1, we can see that k is a constant.

Let M be a complex manifold with a torsionfree affine connection which preserves the almost complex structure tensor J. It is natural to say that M is a *manifold of constant holomorphic curvature* if $R(X, Y)Z$ is a linear combination of X, Y, JX, JY, and JZ.

References

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