# 240. On Integers Expressible as a Sum of Two Powers

By Palahenedi Hewage DIANANDA
Department of Mathematics, University of Singapore, Singapore

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1. In a recent paper [2] Uchiyama proved the following results.

**Theorem 1.** *For every $n\geqq 1$ there are positive integers $x$ and $y$ satisfying*

$$n<x^2+y^2<n+2^{\frac{3}{2}}n^{\frac{1}{4}}.$$

**Theorem 2.** *For any $\varepsilon>0$ there is $n_0=n_0(\varepsilon)$ such that for every $n\geqq n_0$ there are positive integers $x$ and $y$ satisfying*

$$n<x^h+y^h<n+(c+\varepsilon)n^a,$$

*where*

$$a=\left(1-\frac{1}{h}\right)^2,\quad c=h^{2-\frac{1}{h}}\quad \text{and}\quad h\geqq 2.$$

The case $h=2$ is due to Bambah and Chowla [1] and Theorem 1 is a refinement of their result.

It is the main purpose of this note to obtain the following refinement of Theorem 2.

**Theorem 3.** *There is $n_0$ such that for every $n\geqq n_0$ there are positive integers $x$ and $y$ satisfying*

$$n<x^h+y^h<n+cn^a,$$

*where $a$, $c$, and $h$ are as in Theorem 2.*

This will be deduced from the following result.

**Theorem 4.** *Let $h\geqq 2$,*

$$N=N(n)=(n^{1/h}+1)^h-n+1$$

*and*

$$g(n)=N-(N^{1/h}-1)^h.$$

*Then for every $n\geqq 1$ there are positive integers $x$ and $y$ satisfying*

$$n<x^h+y^h<n+g(n).$$

The case $h=2$ of Theorem 3 is weaker than Theorem 1 which however can be obtained easily from Theorem 4.

Extensions of Theorems 4 and 2 and another result of Uchiyama to sums of the form $x^f+y^h$ (in place of $x^h+y^h$) will be given later.

Our proofs have similarities with those of Uchiyama and Bambah and Chowla.

2. We omit the proof of Theorem 4 which is a special case of Theorem 4A to be proved later.

Theorem 3 follows easily from Theorem 4 and the following lemma.

**Lemma 1.** *$g(n) < cn^a$ for large $n$.*

*Proof.* We have

$$N(n) = (n^{1/h}+1)^h - n + 1$$
$$= hn^{1-\frac{1}{h}}\{1+O(n^{-\frac{1}{h}})\},$$

as $n \to \infty$. Hence

$$g(n) = N - (N^{1/h}-1)^h$$
$$= hN^{1-\frac{1}{h}}\left\{1 - \frac{h-1}{2}N^{-\frac{1}{h}} + O(N^{-\frac{2}{h}})\right\}$$
$$= cn^a\left\{1 - \frac{h-1}{2}h^{-\frac{1}{h}}n^{-\frac{h-1}{h^2}} + O(n^{-\frac{1}{h}})\right\},$$

as $n \to \infty$. Hence $g(n) < cn^a$ for large $n$.

*Remark* 1. $g(n) \sim cn^a$ for large $n$.

*Remark* 2. When $h=2$, $N(n) = 2(n^{\frac{1}{2}}+1)$ and so, for $n \geqq 3$, $g(n) = 2^{\frac{3}{2}}(n^{\frac{1}{2}}+1)^{\frac{1}{2}} - 1 < 2^{\frac{3}{2}}n^{\frac{1}{4}}$ since $2^{\frac{3}{2}} < (n^{\frac{1}{2}}+1)^{\frac{1}{2}} + n^{\frac{1}{4}}$. Also $g(n) \approx 3.4 \approx 2^{\frac{3}{2}}n^{\frac{1}{4}}$ for $n=2$ and $g(n)=3$, $2^{\frac{3}{2}}n^{\frac{1}{4}} \approx 2.8$ for $n=1$. Hence Theorem 1 follows easily from Theorem 4.

3. We now generalize Theorems 4 and 2.

**Theorem 4A.** *Let $f$ and $h \geqq 2$,*

$$N = N(n) = (n^{1/f}+1)^f - n + 1$$

*and*

$$g(n) = N - (N^{1/h}-1)^h.$$

*Then for every $n \geqq 1$ there are positive integers $x$ and $y$ satisfying*

$$n < x^f + y^h < n + g(n).$$

*Proof.* Clearly $N$ increases with $n$ and $g(n)$ with $N$. Thus $N \geqq 2^f \geqq 4$ and $g(n) \geqq 4 - (4^{1/h}-1)^h \geqq 3$. Thus the theorem is clearly true if $[n^{1/f}] = n^{1/f}$. In the rest of the proof we therefore assume that

$$m = [n^{1/f}] < n^{1/f}. \tag{1}$$

The theorem is clearly true if

$$(m+1)^f + 1 < n + g(n).$$

In the rest of the proof we therefore assume that

$$(m+1)^f + 1 \geqq n + g(n). \tag{2}$$

Since

$$m^f + \{[(n-m^f)^{1/h}]+1\}^h \leqq m^f + \{(n-m^f)^{1/h}+1\}^h$$

the theorem follows easily from the following lemma.

**Lemma 2.** (1) *and* (2) *imply that*

$$m^f + \{(n-m^f)^{1/h}+1\}^h < n + g(n).$$

*Proof.* From (2)

$$n - m^f \leqq (m+1)^f - m^f + 1 - g(n).$$

Clearly $(m+1)^f - m^f$ increases with $m$. Hence from (1)

$$n-m^f<(n^{1/f}+1)^f-n+1-g(n)$$
$$=N-g(n)=(N^{1/h}-1)^h.$$

Hence

$$m^f+\{(n-m^f)^{1/h}+1\}^h=n+\{(n-m^f)^{1/h}+1\}^h-(n-m^f)$$
$$<n+N-(N^{1/h}-1)^h=n+g(n)$$

since $\{(n-m^f)^{1/h}+1\}^h-(n-m^f)$ clearly increases with $n-m^f$. This completes the proof.

*Remark* 3. In the definitions of $N(n)$ and $g(n)$ in Theorem 4A the roles of $f$ and $h$ can be interchanged to yield a new function $g(n)$.

**Theorem 2A.** *For any $\varepsilon>0$ there is $n_0=n_0(\varepsilon)$ such that for every $n\geqq n_0$ there are positive integers $x$ and $y$ satisfying*

$$n<x^f+y^h<n+(c+\varepsilon)n^a,$$

*where*

$$a=\left(1-\frac{1}{f}\right)\left(1-\frac{1}{h}\right),\quad c=hf^{1-\frac{1}{h}}\quad \textit{and } f \textit{ and } h\geqq 2.$$

This follows from Theorem 4A and the following lemma.

**Lemma 3.** *$g(n)\sim cn^a$ for large $n$.*

*Proof.* We have

$$N(n)=(n^{\frac{1}{f}}+1)^f-n+1\sim fn^{1-\frac{1}{f}}$$

for large $n$. Hence

$$g(n)=N-(N^{\frac{1}{h}}-1)^h\sim hN^{1-\frac{1}{h}}\sim cn^a$$

for large $n$. This completes the proof.

*Remark* 4. In Theorem 2A we can replace $c$ by $C=fh^{1-\frac{1}{f}}$ by interchanging $f$ and $h$. If $f<h$ there is an improvement since $C<c$. If $f>h$ it can be proved that $c+\varepsilon$ can be replaced by $C$ but there is no improvement since $c+\varepsilon<C$ if $\varepsilon$ is small.

**4.** We omit the proof of the following result result.

**Theorem 5.** *For any $\varepsilon>0$ the set of integers $n$ for which the interval $(n, n+\varepsilon n^a)$, where $a, f,$ and $h$ are as in Theorem 2A, contains an integer of the form $x^f+y^h$ has a positive density.*

The case $f=h$ is due to Uchiyama [2]. His proof can be easily modified to prove our result.

## References

[1] R. P. Bambah and S. Chowla: On numbers which can be expressed as a sum of two squares. Proc. Nat. Inst. Sci. India, **13**, 101-103 (1947).

[2] S. Uchiyama: On the distribution of integers representable as a sum of two $h$-th powers. J. Fac. Sic., Hokkaidô Univ., Ser. I, **18**, 124-127(1965).