7. On Hausdorff's Theorem

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In his paper [2], Professor T. Satō considers directed sequences of real numbers, and the Riemann-Stieltjes integral as its application.

In the case of the Riemann-Stieltjes integral, he generalizes Darboux's theorem on the Riemann integral and obtains the following two theorems:

Theorem 1. *Let* $\{\psi_n(x)\}$ *be a sequence of bounded functions in* $[a, b]$.

If $\psi_1(x) \geqq \psi_2(x) \geqq \cdots \geqq \psi_n(x) \geqq \cdots$, *and*

$$\lim_{n\to\infty} \psi_n(x)=0,$$

then

$$\lim_{n\to\infty} \int_a^b \psi_n(x)d\sigma(x)=0.$$

Theorem 2. *Let* $\{f_n(x)\}$ *be a sequence of uniformly bounded functions in* $[a, b]$.

If a sequence of functions $f_n(x)$ $(n=1, 2, \cdots)$ *converges to a function* $f(x)$, *then*

$$\overline{\lim_{n\to\infty}} \underline{\int_a^b} f_n(x)d\sigma(x) \leqq \overline{\int_a^b} f(x)d\sigma(x),$$

$$\underline{\lim_{n\to\infty}} \overline{\int_a^b} f_n(x)d\sigma(x) \geqq \underline{\int_a^b} f(x)d\sigma(x).$$

We shall generalize the latter using his method.

In this note, we shall prove the following theorem which is a generalization of the theorem 2.

Theorem. *Let* $\{f_n(x)\}$ *be a sequence of uniformly bounded functions in* $[a, b]$.

Let $\underline{f}(x)=\underline{\lim}_{n\to\infty} f_n(x)$, $\bar{f}(x)=\overline{\lim}_{n\to\infty} f_n(x)$, *then we have*

$$\overline{\lim_{n\to\infty}} \underline{\int_a^b} f_n(x)d\sigma(x) \leqq \overline{\int_a^b} \bar{f}(x)d\sigma(x),$$

$$\underline{\lim_{n\to\infty}} \overline{\int_a^b} f_n(x)d\sigma(x) \geqq \underline{\int_a^b} \underline{f}(x)d\sigma(x).$$

To prove the theorem above, we shall first explain some notations.

Let $\sigma(x)$ be a continuous and strictly increasing function in $[a, b]$. We subdivide the interval $[a, b]$ by means of the points $x_0, x_1, \cdots, x_{n-1}, x_n$, so that

$$D: a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

We consider a set of subdivisions D and denote it by $\mathfrak{D}$.

Let $m_j, M_j (j=1, 2, \cdots, n)$ be the greatest lower and the least upper bounds of $f(x)$ in the subinterval $[x_{j-1}, x_j]$ respectively. Put

$$s_D(f) = \sum_{j=1}^{n} m_j(\sigma(x_j) - \sigma(x_{j-1})),$$

$$S_D(f) = \sum_{j=1}^{n} M_j(\sigma(x_j) - \sigma(x_{j-1})).$$

Following Darboux terminology, $\sup_{D \in \mathfrak{D}} s_D(f)$ and $\inf_{D \in \mathfrak{D}} S_D(f)$ are called a upper and a lower integrals respectively.

Further we use the following notations.

$$\underline{\int_a^b} f(x) d\sigma(x) = \lim_{D \in \mathfrak{D}} s_D(f),$$

$$\overline{\int_a^b} f(x) d\sigma(x) = \lim_{D \in \mathfrak{D}} S_D(f).$$

Then we have

$$\underline{\int_a^b} f(x) d\sigma(x) = \sup_{D \in \mathfrak{D}} s_D(f),$$

$$\overline{\int_a^b} f(x) d\sigma(x) = \inf_{D \in \mathfrak{D}} S_D(f).$$

Now we shall prove the theorem, mentioned above.

Put

(1) $$\varphi_n(x) = \inf_{n \leqq k} f_k(x).$$

Then $\{\varphi_n(x)\}$ is a monotone non decreasing sequence of bounded functions.

Put $\psi_n(x) = \underline{f}(x) - \varphi_n(x)$. Then $\{\psi_n(x)\}$ is a monotone non increasing sequence of bounded functions and

$$\lim_{n \to \infty} \psi_n(x) = 0.$$

Therefore, by Theorem 1, we have

$$\lim_{n \to \infty} \underline{\int_a^b} \psi_n(x) d\sigma(x) = 0.$$

Hence for every $\varepsilon > 0$ there exists a positive integer N such that

(2) $$\underline{\int_a^b} \psi_n(x) d\sigma(x) < \varepsilon \qquad \text{for} \quad n \geqq N.$$

Let I be any interval contained in $[a, b]$. Then

$$\inf_{x \in I} (\underline{f}(x) - \varphi_n(x)) \geqq \inf_{x \in I} \underline{f}(x) - \sup_{x \in I} \varphi_n(x).$$

Hence

$$s_D(\underline{f} - \varphi_n) \geqq s_D(\underline{f}) - S_D(\varphi_n).$$

Consequently

$$\lim_{D \in \mathfrak{D}} s_D(\underline{f} - \varphi_n) \geqq \lim_{D \in \mathfrak{D}} \{s_D(\underline{f}) - S_D(\varphi_n)\}$$

$$=\lim_{D\in\mathfrak{D}} s_D(\underline{f})-\lim_{D\in\mathfrak{D}} S_D(\varphi_n),$$

which is written to the form of

$$\underline{\int_a^b}(\underline{f}(x)-\varphi_n(x))d\sigma(x)\geqq\underline{\int_a^b}\underline{f}(x)d\sigma(x)-\overline{\int_a^b}\varphi_n(x)d\sigma(x).$$

By the inequality (2), we have

$$\underline{\int_a^b}(\underline{f}(x)-\varphi_n(x))d\sigma(x)=\underline{\int_a^b}\psi_n(x)d\sigma(x)<\varepsilon \qquad \text{for} \quad n\geqq N.$$

Hence

$$\underline{\int_a^b}\underline{f}(x)d\sigma(x)<\overline{\int_a^b}\varphi_n(x)d\sigma(x)+\varepsilon \qquad \text{for} \quad n\geqq N.$$

By (1), we have

$$\varphi_n(x)\leqq f_n(x) \qquad (n=1, 2, \cdots).$$

Therefore

$$\overline{\int_a^b}\varphi_n(x)d\sigma(x)\leqq\overline{\int_a^b}f_n(x)d\sigma(x).$$

Hence

$$\underline{\int_a^b}\underline{f}(x)d\sigma(x)<\overline{\int_a^b}f_n(x)d\sigma(x)+\varepsilon \qquad \text{for} \quad n\geqq N.$$

Since ε is arbitrary, it follows that

$$\underline{\int_a^b}\underline{f}(x)\, d\sigma(x)\leqq\varliminf_{n\to\infty}\overline{\int_a^b}f_n(x)d\sigma(x),$$

and similarly

$$\varlimsup_{n\to\infty}\underline{\int_a^b}f_n(x)d\sigma(x)\leqq\overline{\int_a^b}\bar{f}(x)d\sigma(x).$$

Remark 1. If $\sigma(x)\equiv x$, then we have the case given by F. Hausdorff [1].

Remark 2. In the Theorem, if a sequence of functions $f_n(x)$ $(n=1, 2, \cdots)$ converges to a function $f(x)$, then we obtain the Theorem 2.

References

[1] F. Hausdorff: Beweis eines Satzes von Arzelà. Math. Zeit., **26**, 135–137 (1927).

[2] T. Satō: Sur l'analyse générale V (Théorie des suits filtrantes de nombres). Annali di Math. Pura ed Appl. (in press).