2. *Weak Topologies and Injective Modules*

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Wu showed in his paper [1] that a characterization of self-injective rings can be given in terms of weak topologies. The aim of this paper is to generalize this result and give a characterization of injective modules.

Throughout this paper, R will denote a ring (not necessarily commutative) with the identity 1. All R-modules considered will be unitary. We shall show that, if an R-module Q has the property: $\operatorname{ann} Q=0$, where $\operatorname{ann} Q$ denotes the ideal of R consisting of all elements annihilating Q, then a necessary and sufficient condition for the module Q to be injective can be given in terms of weak topologies. In addition to it, we shall show at the end of this paper, that such a simple generalization of the theorem due to Wu is not always available in case $\operatorname{ann} Q\neq 0$.

1. Weak topologies. Let Q be a left R-module. Then $\operatorname{Hom}_R(Q, Q)$ can be regarded as a ring with the identity ι_Q. Λ will denote this ring $\operatorname{Hom}_R(Q, Q)$.

Let M be a left R-module. Then $\operatorname{Hom}_R(M, Q)$ can be regarded as a left Λ-module, since we have $\varphi\circ\rho\in\operatorname{Hom}_R(M, Q)$ for any $\varphi\in\Lambda$ and for any $\rho\in\operatorname{Hom}_R(M, Q)$.

Now we shall give the module Q a structure of topological space with the discrete topology. In connexion with this topology, we shall give the following definition of B-topology on a module M. (cf. Chase [2])

Definition. Let B be a Λ-submodule of the left Λ-module $\operatorname{Hom}_R(M, Q)$. Then the coarsest topology on M such that every element of B is a continuous mapping from M into Q will be called the *weak topology on M induced by B* or simply the *B-topology on M*.

It is easy to see that all subsets of M of the form $\bigcap_{i=1}^{n}\operatorname{Ker}\beta_i$, $\beta_i\in B$, $i=1, 2, \cdots, n$ make a base of neighbourhood system of $0(\in M)$ in the B-topology.

It is obvious that the B-topology on M is Hausdorff if and only if, for each non-zero $x(x\in M)$, there exists $\beta(\beta\in B)$ such that $x\notin \operatorname{Ker}\beta$. According to Wu, we shall say B is *separating* if the weak topology on M induced by B is Hausdorff.

It is evident that $\operatorname{Hom}_R(Q, Q)$-topology on Q is the discrete topology since $\mathrm{Hom}_R(Q, Q)$ includes the identity mapping ι_Q.

Theorem 1. Let $C_B\,\mathrm{Hom}_R(M, Q)$ be a subset of $\mathrm{Hom}_R(M, Q)$ consisting of all continuous R-homomorphisms from M with the B-topology into Q with the discrete topology. Then $C_B\,\mathrm{Hom}_R(M, Q)$ is a left Λ-submodule of the module $\mathrm{Hom}_R(M, Q)$.

Proof. It is evident that the $C_B\,\mathrm{Hom}_R(M, Q)$ is a left Z-submodule of the Z-module $\mathrm{Hom}_R(M, Q)$. Hence it is sufficient to prove that $\varphi\circ C_B\,\mathrm{Hom}_R(M, Q)\subset C_B\,\mathrm{Hom}_R(M, Q)$ for any $\varphi\in\mathrm{Hom}_R(Q, Q)$. Let ρ be an arbitrary element of $C_B\,\mathrm{Hom}_R(M, Q)$, then, by the definition of the B-topology, there exist $\beta_1, \beta_2, \cdots, \beta_n$, $\beta_i\in B$, $i=1, 2, \cdots, n$ such that $\mathrm{Ker}\,\rho\supset\bigcap_{i=1}^n\mathrm{Ker}\,\beta_i$. Since $\mathrm{Ker}\,(\varphi\circ\rho)\supset\mathrm{Ker}\,\rho$, it is easy to see that $\mathrm{Ker}\,(\varphi\circ\rho)\supset\bigcap_{i=1}^n\mathrm{Ker}\,\beta_i$. Therefore we have proved that $\varphi\circ\rho$ is a continuous mapping, i.e. an element of $C_B\,\mathrm{Hom}_R(M, Q)$.

In case $Q=R$, we can define the anti-isomorphism t from the ring $\mathrm{Hom}_R(R, R)$ onto the ring R itself by $t(\varphi)=\varphi(1)$, $\varphi\in\mathrm{Hom}_R(R, R)$. Let M be a left R-module, then it is easy to verify that $\mathrm{Hom}_R(M, R)$ can be regarded as a right R-module. After these considerations, we have immediately the following corollary to the Theorem 1:

Corollary. Let B be a right R-submodule of the module $\mathrm{Hom}_R(M, R)$. Then $C_B\,\mathrm{Hom}_R(M, R)$ is a right R-submodule of the module $\mathrm{Hom}_R(M, R)$.

The following theorem is a main theorem of this paper:

Theorem 2. Let Q be a finitely generated left R-module. Suppose $\mathrm{ann}\,Q=0$. Then the following four statements (i)$\sim$(iv) are equivalent:

(i) Q is left R-injective.

(ii) $B=C_B\,\mathrm{Hom}_R(M, Q)$ for any left R-module M (not necessarily finitely generated) and any Λ-submodule B of the Λ-module $\mathrm{Hom}_R(M, Q)$.

(iii) $B=C_B\,\mathrm{Hom}_R(M, Q)$ for any left R-module M and any separating Λ-submodule B of the Λ-module $\mathrm{Hom}_R(M, Q)$.

(iv) $B=C_B\,\mathrm{Hom}_R(I, Q)$ for any left ideal I of R and any separating Λ-submodule B of the Λ-module $\mathrm{Hom}_R(I, Q)$.

Proof. (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(iv) are self-evident.

(iv)$\Rightarrow$(i) will be proved as follows: Suppose that Q is not left R-injective. Then there exists a left ideal I of R such that the homomorphism j^*: $\mathrm{Hom}_R(R, Q)\to\mathrm{Hom}_R(I, Q)$ induced by the canonical injection j: $I\to R$ is not surjective. (See Cartan-Eilenberg, Homological Algebra, p. 8.)

Set $B=j^*(\mathrm{Hom}_R(R, Q))$, then it is easy to see that this B is a Λ-submodule of the Λ-module $\mathrm{Hom}_R(I, Q)$.

Let $Q=Rg_1+Rg_2+\cdots+Rg_s$, and let $h_1, h_2, \cdots, h_s$ be R-homomorphism from R into Q defined by $h_i(1)=g_i$, $i=1, 2, \cdots, s$. It is easy to see that $\bigcap_{i=1}^{s} \operatorname{Ker} j^*(h_i)=I\cap \operatorname{ann} Q$. Since $j^*(h_i)\in B$, and since we have assumed that $\operatorname{ann} Q=0$, we can conclude immediately that the B-topology on I is the discrete topology. Therefore it is obvious that every element of $\operatorname{Hom}_R(I, Q)$ is a continuous mapping from I into Q. Since j^* is not a surjection, we have immediately the following strict inclusion: $B(=j^*(\operatorname{Hom}_R(R, Q)))\subsetneqq C_B \operatorname{Hom}_R(I, Q)$ $(=\operatorname{Hom}_R(I, Q))$. Thus we have proved (iv)⇒(i).

The proof of (i)⇒(ii) is quite similar to the proof which is given by Wu in his paper [1]. Suppose that Q is left R-injective, and let M be a left R-module. Let B be a Λ-submodule of the module $\operatorname{Hom}_R(M, Q)$, and let ρ be an arbitrary element of $C_B \operatorname{Hom}_R(M, Q)$. In order to complete the proof, we have only to show that ρ is contained in B.

Since we have assumed that ρ is a continuous mapping, there exist $\beta_1, \beta_2, \cdots, \beta_n$, $\beta_i\in B$, $i=1, 2, \cdots, n$ such that $\operatorname{Ker}\rho\supset\bigcap_{i=1}^{n}\operatorname{Ker}\beta_i$. Let $U=Q\oplus Q\oplus\cdots\oplus Q$ be a direct sum of n copies of the module Q, and let $f\colon M\to U$ be the R-homomorphism defined by $f(x)=(\beta_1(x), \beta_2(x), \cdots, \beta_n(x))$ for $x\in M$. Then it is evident that $\operatorname{Ker} f=\bigcap_{i=1}^{n}\operatorname{Ker}\beta_i$. Now let $j\colon \operatorname{Im} f\to U$ be the canonical injection, and let $g: M\to\operatorname{Im} f$ be the R-homomorphism such that $j\circ g=f$. Since $\operatorname{Ker} g\subset\operatorname{Ker}\rho$, and since g is an epimorphism, it is easy to see that there exists an R-homomorphism $\sigma\colon \operatorname{Im} f\to Q$ such that $\sigma\circ g=\rho$. From the assumption that Q is left R-injective, it is easily seen that the homomorphism σ can be extended to an R-homomorphism $\tau\colon U\to Q$ such that $\tau\circ f=\rho$.

Let $\tau_i\colon Q\to Q$ be the i-th component of $\tau\colon U\to Q$. Then we have $\tau(y)=\sum_{i=1}^{n}\tau_i(y)$ for $y\in U$. Using above results, we have immediately the following equation for any $x\in M$:

$$\rho(x)=\tau\circ f(x)=\sum_{i=1}^{n}\tau_i(\beta_i(x)).$$

This equation shows that $\rho=\sum_{i=1}^{n}\tau_i\circ\beta_i$.

Since we have assumed that B is a Λ-submodule of the module $\operatorname{Hom}_R(M, Q)$, and since $\tau_i\in\Lambda$, $i=1, 2, \cdots, n$, it is obvious that $\rho\in B$. Thus we have proved (i)⇒(ii).

2. The case $\operatorname{ann} Q\neq 0$. In case $\operatorname{ann} Q\neq 0$, the conclusion of the theorem 2 above stated does not always hold. To make clear this situation, we shall give an example of a ring R and an R-module Q with the following properties: (a) $\operatorname{ann} Q\neq 0$, (b) Q is not R-injective, (c) $B=C_B \operatorname{Hom}_R(M, Q)$ for any R-module M and for any Λ-submodule B of the module $\operatorname{Hom}_R(M, Q)$.

Example. Let R be a primary local ring with the maximal ideal $\mathfrak{m}=(u)$, where $u^2=0$. (a) Then it is evident that $\operatorname{ann}\mathfrak{m}=\mathfrak{m}\neq 0$. (b) It is also easy to see that $\mathfrak{m}$ is not injective as an R-module. (Actually it is easy to see that the essential injective envelope of the R-module $\mathfrak{m}$ is R itself.)

(c) Hereafter the ideal $\mathfrak{m}$ will be understood as a topological space with the discrete topology. Let $F=R/\mathfrak{m}$ be the residue field, and $f: R\to F$ the canonical epimorphism. Let M be an R-module (not necessarily finitely generated). Since $\operatorname{ann}\mathfrak{m}=\mathfrak{m}$, it is easy to see that $\operatorname{Hom}_R(M, \mathfrak{m})$ can be regarded as an F-module (i.e. a vector space over F of finite or infinite dimension). It is also obvious that the ring $\operatorname{Hom}_R(\mathfrak{m}, \mathfrak{m})$ is isomorphic to F.

Let B be an F-submodule of the F-module $\operatorname{Hom}_R(M, \mathfrak{m})$. Let $C_B\operatorname{Hom}_R(M, \mathfrak{m})$ be the set consisting of all continuous homomorphisms from the module M with the B-topology into $\mathfrak{m}$ with the discrete topology. Let ρ be an arbitrary element of $C_B\operatorname{Hom}_R(M, \mathfrak{m})$. For the present purpose, it will be sufficient to prove that $\rho\in B$.

By the definition of the B-topology, there exist $\beta_1, \beta_2, \cdots, \beta_n$, $\beta_i\in B$, $i=1, 2, \cdots, n$ such that $\operatorname{Ker}\rho\supset\bigcap_{i=1}^n\operatorname{Ker}\beta_i$. Since $\operatorname{Hom}_R(M, \mathfrak{m})$ is a vector space, we can choose elements $\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_k}$ from $\beta_1, \beta_2, \cdots, \beta_n$ in such a way that these $\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_k}$ are linearly independent over F and any of $\beta_1, \beta_2, \cdots, \beta_n$ can be expressed as a linear combination of $\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_k}$. Then it is easy to see that $\bigcap_{i=1}^n\operatorname{Ker}\beta_i=\bigcap_{t=1}^k\operatorname{Ker}\beta_{i_t}$. Therefore it is evident that we can assume without any loss of generality that $\beta_1, \beta_2, \cdots, \beta_n$ are linearly independent over F.

Let $\beta_i(a_\mu)=a_{i\mu}u$, where $\{a_\mu\}_{\mu\in\Gamma}$ be an R-basis (not necessarily minimal) of the R-module M. Let $\bar{a}_{i\mu}=f(a_{i\mu})$, where f is the canonical epimorphism from the ring R onto the field F. Let W be the linear subspace spanned by all column vectors of the type

$$\begin{bmatrix}\bar{a}_{1\mu}\\ \bar{a}_{2\mu}\\ \vdots\\ \bar{a}_{n\mu}\end{bmatrix}, \mu\in\Gamma.$$

Then it is self-evident that $\dim W\leqq n$, and it is easy to see that $\dim W$ is just equal to n. Now let $b_1, b_2, \cdots, b_n$ be elements chosen from $\{a_\mu\}_{\mu\in\Gamma}$ in such a way that $\det(\bar{b}_{ij})\neq 0$ where $\beta_i(b_j)=b_{ij}u$, $\bar{b}_{ij}=f(b_{ij})$. Let $(\bar{c}_{ij})$ be the inverse matrix of the matrix $(\bar{b}_{ij})$, and c_{ij} be a representative of each of $\bar{c}_{ij}$ such that $f(c_{ij})=\bar{c}_{ij}$, $i=1, 2, \cdots, n$, $j=1, 2, \cdots, n$. Let $c_j=\sum_{k=1}^n c_{kj}b_k$, then we have $\beta_i(c_j)=\sum_{k=1}^n c_{kj}\beta_i(b_k)=\sum_{k=1}^n c_{kj}b_{ik}u$. From the definition of $\bar{c}_{kj}$, we have $\sum_{k=1}^n \bar{b}_{ik}\bar{c}_{kj}=\delta_{ij}$. Therefore we have immediately $\beta_i(c_j)=\delta_{ij}u$ since $u^2=0$.

Let $d_\tau=a_\tau-\sum_{j=1}^n a_{j\tau}c_j$, $\tau\in\Gamma$. Then it is easy to see that $c_1, c_2,$

$\cdots, c_n$ and $\{d_\tau\}_{\tau\in\Gamma}$ generate the R-module M.

From the definition of $\{d_\tau\}_{\tau\in\Gamma}$, we have immediately

$$\beta_i(d_\tau)=\beta_i(a_\tau)-\sum_{j=1}^n a_{j\tau}\beta_i(c_j)=a_{i\tau}u-a_{i\tau}u=0.$$

This implies that $d_\tau \in \bigcap_{i=1}^n \operatorname{Ker}\beta_i \subset \operatorname{Ker}\rho$.

Let $\rho(c_i)=h_iu$, $i=1, 2, \cdots, n$, and let $\sigma=\rho-\sum_{j=1}^n h_j\beta_j$. Then we have $\sigma(c_i)=\rho(c_i)-\sum_{j=1}^n h_j\beta_j(c_i)=h_iu-h_iu=0$, $i=1, 2, \cdots, n$. On the other hand, it is obvious that $\sigma(d_\tau)=0$, $\tau\in\Gamma$, since $d_\tau\in\operatorname{Ker}\rho$ and $d_\tau\in\operatorname{Ker}\beta_i$, $i=1, 2, \cdots, n$. Since $c_1, c_2, \cdots, c_n$ and $\{d_\tau\}_{\tau\in\Gamma}$ make an R-basis of the R-module M, we have immediately $\sigma=0$, i.e. $\rho=\sum_{j=1}^n h_j\beta_j$. Thus we have proved that ρ belongs to B.

References

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