22. *On a Sum Theorem in Dimension Theory*

By Nobuo KIMURA

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The present paper deals primarily with the sum theorems for the large inductive dimension of totally normal spaces.[1] In this connection C. H. Dowker established in [2] a sum theorem which is stated as follows: *Let $\{A_i\}$ be a countable number of closed sets in a totally normal space and let* Ind A_i[2] $\leqq n$, $i=1, 2, \cdots$. *Then* Ind $\left(\bigcup_{i=1}^{\infty} A_i\right) \leqq n$. Corresponding to this result, we established in [3] the following theorem. *Let $\{A_\alpha \mid \alpha<\Omega\}$ be a locally finite closed covering of a totally normal and countably paracompact space X and let* Ind $A_\alpha \leqq n$ *for each α. Then* Ind $X \leqq n$.

Our present object is to show that the countable paracompactness condition in the above theorem is redundant. Indeed, our main theorem reads as follows: *Let $\{A_\alpha \mid \alpha<\Omega\}$ be a locally finite collection of closed sets in a totally normal space X and let* Ind $A_\alpha \leqq n$ *for each α. Then* Ind $\left(\bigcup_{\alpha<\Omega} A_\alpha\right) \leqq n$. For the proof of our theorems we shall need some of Dowker's results.

1. Preliminary theorems due to C. H. Dowker. A normal space X is called *totally normal* ([2, § 4]) if each open set G is the union of a collection $\{G_\alpha\}$, locally finite in G, of open F_σ sets of X. The following theorems are due to C. H. Dowker and they form the basis of a proof for our theorems.

Theorem 1. ([2, 4.1], [2, 4.2], and [2, 4.6]). *Every perfectly normal space or every hereditarily paracompact space is totally normal and every totally normal space is completely normal.*[3]

The converse of Theorem 1 is not true as is observed by the well-known Bing's examples ([1]).

Theorem 2. ([2, 4.7]). *The total normality is hereditary; that is, every subspace of a totally normal space is also totally normal.*

Theorem 3. ([2, Theorem 2]). *In a totally normal space X let $A\subset X$. Then* Ind $A \leqq$ Ind X.

Theorem 3 is referred to as "the subset theorem".

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1) Throughout the paper by a *space* we mean a T_1-space.

2) Ind X means the *large inductive dimension* of a space X defined inductively in terms of closed sets. For a detailed definition, see [2].

3) Some authors refer to "completely normal" as "hereditarily normal" (e.g. [5]).

Theorem 4. ([2, 2.2]). *If A is a closed subset of a completely normal space X and if* $\operatorname{Ind} A \leqq n$ *and* $\operatorname{Ind}(X-A) \leqq n$, *then* $\operatorname{Ind} X \leqq n$.

2. Theorems. In this section we list our theorems and their corollaries the proofs of which will be given at section 3.

Theorem 5. *Let $\{A_\alpha \mid \alpha<\Omega\}$ be a locally finite closed covering of a totally normal space X and let* $\operatorname{Ind} A_\alpha \leqq n$ *for each α. Then* $\operatorname{Ind} X \leqq n$.

If X is a hereditarily paracompact space, we have more generally the following theorem which is a generalization of [4, Theorem 5, 2].

Theorem 6. *Let $\{A_\alpha \mid \alpha<\Omega\}$ be a locally countable closed covering of a hereditarily paracompact space X and let* $\operatorname{Ind} A_\alpha \leqq n$ *for each α. Then* $\operatorname{Ind} X \leqq n$.

In view of Theorems 2 and 3, the following is a direct consequence of Theorem 5.

Corollary 1. *Let $\{A_\alpha \mid \alpha<\Omega\}$ be a locally finite collection of closed sets in a totally normal space X and let* $\operatorname{Ind} A_\alpha \leqq n$ *for each α. Then* $\operatorname{Ind} \sum_{\alpha<\Omega}$[4] $A_\alpha \leqq n$.

From Corollary 1 and from Dowker's countable sum theorem we obtain

Theorem 7. *Let $\{A_{i\alpha} \mid \alpha<\Omega, i=1, 2, \cdots\}$ be a σ-locally finite collection of closed sets in a totally normal space X and let* $\operatorname{Ind} A_{i\alpha} \leqq n$ *for each i and α. Then* $\operatorname{Ind} \sum_{i=1}^{\infty} \sum_{\alpha<\Omega} A_{i\alpha} \leqq n$.

By virtue of Theorem 1, as an immediate consequence of Theorem 7, we obtain

Corollary 2. *Let $\{A_{i\alpha} \mid \alpha<\Omega, i=1, 2, \cdots\}$ be a σ-locally finite collection of closed sets in a perfectly normal or hereditarily paracompact space and let* $\operatorname{Ind} A_{i\alpha} \leqq n$ *for each i and α. Then* $\operatorname{Ind} \sum_{i=1}^{\infty} \sum_{\alpha<\Omega} A_{i\alpha} \leqq n$.

3. Proof of Theorem 5. We proceed by induction on n. Since the theorem is trivially true for $n=-1$, we have only to verify it for n assuming it true for $k<n$. Let $F \subset G$ with F closed and G open. To complete our induction we should find an open set W of X such that $F \subset W \subset G$ and $\operatorname{Ind} \mathfrak{B} W$[5] $\leqq n-1$. Let $F_\alpha = F \cdot A_\alpha$ and $G_\alpha = G \cdot A_\alpha$. Then F_α and G_α are closed and open, in A_α, respectively, and $F_\alpha \subset G_\alpha$. All F_α will be assumed to be non-empty without loss of generality. In what follows, "A_α-open" and "A_α-closed" are used in place of "open in A_α" and "closed in A_α" for the sake of simplicity. Now suppose that for every $\beta<\alpha$ an A_β-open set W_β has been so

4) We use frequently the symbols "$\cdot$" and "$+$" instead of "$\cap$" and "$\cup$", respectively.

5) $\mathfrak{B} W$ stands for the boundary of W, $\mathfrak{B} W = \overline{W} - \operatorname{int} W$.

constructed that

(1) (i) $\operatorname{Ind}\mathfrak{B}_\beta W_\beta$[6] $\leqq n-1$,

(ii) $F_\beta\subset W_\beta\subset G_\beta$,

(iii) $W_\beta\cdot A_\gamma=W_\gamma\cdot A_\beta$ for every $\gamma<\alpha$.

Since $\bar{W}_\beta\cdot A_\beta=\bar{W}_\beta$ by virtue of the closedness of A_β, we obtain $\mathfrak{B}_\beta W_\beta=\bar{W}_\beta-W_\beta$. From A_β-closedness of $\mathfrak{B}_\beta W_\beta$ it follows that $\mathfrak{B}_\beta W_\beta$ and hence $\sum_{\beta<\alpha}\mathfrak{B}_\beta W_\beta$ are closed in X. Let $A_\alpha^0=A_\alpha-\sum_{\beta<\alpha}\mathfrak{B}_\beta W_\beta$. Then by (iii) $A_\alpha\cdot\sum_{\beta<\alpha}W_\beta$ is A_α^0-closed. Hence $A_\alpha\cdot\sum_{\beta<\alpha}W_\beta+F_\alpha$ is A_α^0-closed. On the other hand, we obtain $A_\alpha\cdot\sum_{\beta<\alpha}W_\beta+\left(G_\alpha-\sum_{\beta<\alpha}A_\beta\right)$ is A_α^0-open. In fact, since $G_\alpha-\sum_{\beta<\alpha}A_\beta$ is naturally A_α^0-open, it suffices to show that every point $x\in A_\alpha\cdot\sum_{\beta<\alpha}W_\beta$ is an A_α^0-inner point of the given set. For this purpose let $A_{\beta_1}, A_{\beta_2}, \cdots, A_{\beta_k}$ with each $\beta_i<\alpha$ be all the sets which contain x. Then x has a neighborhood $V(x)$ such that $V(x)\cdot\sum\{A_\beta\mid\beta\neq\beta_i,\ i=1,2,\cdots,k,\ \beta<\alpha\}=0$. By means of (1) (iii) we can obtain $x\in W_{\beta_1}\cdot W_{\beta_2}\cdot\cdots\cdot W_{\beta_k}$. Hence x has a neighborhood $U(x)$ such that $U(x)\cdot A_{\beta_i}\subset W_{\beta_i}$, $i=1,2,\cdots,k$. Since $x\in\sum_{\beta<\alpha}W_\beta\subset G$, we have $x\in A_\alpha\cdot G=G_\alpha$. Again x has a neighborhood $W(x)$ such that $A_\alpha\cdot W(x)\subset G_\alpha$. Let $N(x)=V(x)\cdot U(x)\cdot W(x)$. From the definition of $V(x)$, $U(x)$, and $W(x)$ it readily follows that $A_\alpha^0\cdot N(x)\subset A_\alpha\cdot N(x)\subset A_\alpha\cdot\sum_{\beta<\alpha}W_\beta+\left(G_\alpha-\sum_{\beta<\alpha}A_\beta\right)$ and this shows that $A_\alpha\cdot\sum_{\beta<\alpha}W_\beta+\left(G_\alpha-\sum_{\beta<\alpha}A_\beta\right)$ is A_α^0-open. Since $A_\alpha\cdot\sum_{\beta<\alpha}W_\beta+F_\alpha\subset A_\alpha\cdot\sum_{\beta<\alpha}W_\beta+\left(G_\alpha-\sum_{\beta<\alpha}A_\beta\right)$ and $\operatorname{Ind}A_\alpha^0\leqq\operatorname{Ind}A_\alpha\leqq n$, there is an A_α^0-open set W_α such that

(2) (i) $A_\alpha\cdot\sum_{\beta<\alpha}W_\beta+F_\alpha\subset W_\alpha\subset A_\alpha\cdot\sum_{\beta<\alpha}W_\beta+\left(G_\alpha-\sum_{\beta<\alpha}A_\beta\right)$,

(ii) $\operatorname{Ind}\mathfrak{B}_{\alpha_0}W_\alpha\leqq n-1$,

where $\mathfrak{B}_{\alpha_0}W_\alpha=\bar{W}_\alpha\cdot A_\alpha^0-W_\alpha$. This W_α satisfies that

(3) (i) $F_\alpha\subset W_\alpha\subset G_\alpha$ and W_α is A_α-open,

(ii) $\operatorname{Ind}\mathfrak{B}_\alpha W_\alpha\leqq n-1$,

(iii) for every $\gamma\leqq\alpha$ $W_\gamma\cdot A_\alpha=W_\alpha\cdot A_\gamma$.

The proof of (3) (i) is immediate from the fact that an A_α^0-open set is at the same time an A_α-open set. (3) (ii) is shown as follows. By calculation we have $\mathfrak{B}_\alpha W_\alpha=\mathfrak{B}_\alpha W_\alpha\cdot A_\alpha=\mathfrak{B}_\alpha W_\alpha\cdot\left(A_\alpha^0+\sum_{\beta<\alpha}\mathfrak{B}_\beta W_\beta\right)$ $=(\bar{W}_\alpha-W_\alpha)\cdot A_\alpha^0+\left(\mathfrak{B}_\alpha W_\alpha\cdot\sum_{\beta<\alpha}\mathfrak{B}_\beta W_\beta\right)=\mathfrak{B}_{\alpha_0}A_\alpha+\mathfrak{B}_\alpha W_\alpha\cdot\sum_{\beta<\alpha}\mathfrak{B}_\beta W_\beta$. By the subset theorem and the induction hypothesis we obtain $\operatorname{Ind}\left(\mathfrak{B}_\alpha W_\alpha\cdot\sum_{\beta<\alpha}\mathfrak{B}_\beta W_\beta\right)\leqq\operatorname{Ind}\sum_{\beta<\alpha}\mathfrak{B}_\beta W_\beta\leqq n-1$. On the other hand, since $\mathfrak{B}_\alpha W_\alpha\cdot\sum_{\beta<\alpha}\mathfrak{B}_\beta W_\beta$ is closed in X, it is a priori closed in $\mathfrak{B}_\alpha W_\alpha$. Since

6) $\mathfrak{B}_\beta W_\beta$ means A_β-*boundary*; i.e., $\mathfrak{B}_\beta W_\beta=\bar{W}_\beta\cdot A_\beta-W_\beta$ in view of A_β-openness of W_β. Notice that in general $\mathfrak{B}W_\beta\neq\mathfrak{B}_\beta W_\beta$.

$\mathfrak{B}_\alpha W_\alpha - \mathfrak{B}_\alpha W_\alpha \cdot \sum_{\beta<\alpha} \mathfrak{B}_\beta W_\beta \subset \mathfrak{B}_{\alpha_0} W_\alpha$, Theorem 4 is applicable and we obtain $\operatorname{Ind} \mathfrak{B}_\alpha W_\alpha = \operatorname{Ind}\left(\mathfrak{B}_{\alpha_0} W_\alpha + \mathfrak{B}_\alpha W_\alpha \cdot \sum_{\beta<\alpha} \mathfrak{B}_\beta W_\beta\right) \leqq n-1$. This proves (3) (ii). Now (3) (iii) remains to be shown. First, by (2) (i), $W_\gamma \cdot A_\alpha \subset W_\alpha \cdot A_\gamma$ is obvious. Hence we have only to prove the converse, $W_\gamma \cdot A_\alpha \supset W_\alpha \cdot A_\gamma$. Since from (2) (i) again it follows that $W_\alpha \cdot A_\gamma \subset A_\alpha \cdot \sum_{\beta<\alpha} W_\beta$, any point $x \in W_\alpha \cdot A_\gamma$ is contained in W_β for some $\beta < \alpha$. For this β we have $x \in A_\gamma \cdot W_\beta$. However, by virtue of (1) (iii), we have $A_\gamma \cdot W_\beta = A_\beta \cdot W_\gamma$, and hence we get $x \in A_\beta \cdot W_\gamma \subset W_\gamma$. Therefore $x \in W_\gamma \cdot W_\alpha \subset W_\gamma \cdot A_\alpha$ and this shows that $W_\gamma \cdot A_\alpha \supset W_\alpha \cdot A_\gamma$. This completes the proof of (3) (iii). By transfinite induction we get finally

Lemma. For any $\alpha < \Omega$ there is an A_α-open set W_α such that
(i) $F_\alpha \subset W_\alpha \subset G_\alpha$,
(ii) $\operatorname{Ind} \mathfrak{B}_\alpha W_\alpha \leqq n-1$,
(iii) $W_\alpha \cdot A_\beta = W_\beta \cdot A_\alpha$ for every $\beta < \Omega$.

We now turn to the proof of Theorem 5. Let $W = \sum_{\alpha<\Omega} W_\alpha$. We shall assert that the set W just defined is actually an open set as desired at the beginning of this section. First, to show the openness of W, let $x \in W$. Since all the A_α's which contain x are at most finite in number, there are $\alpha_1 < \alpha_2 < \cdots < \alpha_k$ such that $x \in A_{\alpha_i}$, $i=1, 2, \cdots, k$ and $x \notin A_\alpha$ otherwise. Since $\sum \{A_\alpha \mid \alpha \neq \alpha_1, \alpha_2, \cdots, \alpha_k\}$ is closed, x has a neighborhood $V(x)$ such that $V(x) \cdot \sum \{A_\alpha \mid \alpha \neq \alpha_1, \alpha_2, \cdots, \alpha_k\} = 0$. While (iii) in the above lemma shows that $W_{\alpha_1}, W_{\alpha_2}, \cdots, W_{\alpha_k}$ and only these contain x. On account of the A_{α_i}-openness of W_{α_i} we can choose, in X, a neighborhood $U_i(x)$ of x so that $U_i(x) \cdot A_{\alpha_i} \subset W_{\alpha_i}$. Let $W(x) = V(x) \cdot U_1(x) \cdot U_2(x) \cdot \cdots \cdot U_k(x)$. Then $W(x) \subset W$. In fact, if some $y \in W(x)$ were not in W, y would belong to A_γ for some γ with $\gamma \neq \alpha_i$, $i=1, 2, \cdots, k$. This is impossible since $V(x) \cdot A_\gamma = 0$ by such γ. Hence W is open. There remains only to prove $\operatorname{Ind} \mathfrak{B} W \leqq n-1$. For this purpose we first prove $\mathfrak{B} W \subset \sum_{\alpha<\Omega} \mathfrak{B}_\alpha W_\alpha$. This is shown as follows:

$$\mathfrak{B} W = \overline{\sum_{\alpha<\Omega} W_\alpha} - \operatorname{int} \sum_{\alpha<\Omega} W_\alpha = \sum_{\alpha<\Omega} \bar{W}_\alpha - \sum_{\alpha<\Omega} W_\alpha \subset \sum_{\alpha<\Omega} (\bar{W}_\alpha - W_\alpha) = \sum_{\alpha<\Omega} \mathfrak{B}_\alpha W_\alpha.$$

By the subset theorem we obtain $\operatorname{Ind} \mathfrak{B} W \leqq \operatorname{Ind} \sum_{\alpha<\Omega} \mathfrak{B}_\alpha W_\alpha$. Now the inequality $\operatorname{Ind} \sum_{\alpha<\Omega} \mathfrak{B}_\alpha W_\alpha \leqq n-1$ follows from the induction hypothesis. This completes the proof of Theorem 5.

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