71. A Remark on Ikegami's Paper "On the Non-Minimal Martin Boundary Points"

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In the theory of the Martin compactification (see [2]), it is of interest to know how many non-minimal points there are. Recently Ikegami ([1]) has proved that, if the set of non-minimal points is not void, it contains infinitely many points.

Let Ω be a Green space, $\hat{\Omega}$ its Martin compactification, Δ the Martin boundary of Ω and Δ_1, Δ_0 the minimal and non-minimal part of Δ respectively.

We will improve Ikegami's result as follows:

Theorem. *If Δ_0 is not void, then Δ_0 is uncountable.*

Proof. Let ω be an open set of Ω, $\{x_n\}$ a sequence of points in ω tending to x_0 of Δ, $\mathcal{E}^{\omega}_{K_{x_n}}(y)$ the extremisation of $K(x_n, y)$ relative to ω, which is written by

$$\mathcal{E}^{\omega}_{K_{x_n}}(y)=\int K(x, y)d\mu_n(x)$$

where $K(x, y)$ is the Martin kernel, μ_n is a positive mass-distribution on $\overset{*}{\omega}\cap\Omega$ and the total mass of μ_n does not exceed 1, $\overset{*}{\omega}$ being the boundary of ω in $\hat{\Omega}$. A subsequence of $\{\mu_n\}$ converges vaguely to μ whose carrier is contained in $\overline{\overset{*}{\omega}\cap\Omega}$.

Clearly,

$$v(y)=\int K(x, y)d\mu(x)$$

is a positive superharmonic function in Ω. Ikegami proved in [1] that

$$\mathcal{E}^{\omega}_{K_{x_0}}(y)\leqq v(y).$$

Let μ_1 be the restriction of μ to Δ, and

$$u(y)=\int K(x, y)d\mu_1(x).$$

Then $u(y)$ is the greatest harmonic minorant of $v(y)$.

Let x_0 be a point of Δ_0. By the Martin representation theorem ([2]), there exists a measure ν on Δ_1 such that

$$K(x_0, y)=\int_{\Delta_1} K(x, y)d\nu(x).$$

We put $D_r=\{x; \rho(x_0, x)<r\}$ and $C_r=\{x; \rho(x_0, x)=r\}$ where ρ implies the Martin metric in $\hat{\Omega}$. There exists an r_0 such that for all r

positive less than r_0

$$\nu(\Delta_1 \cap C\bar{D}_r) > 0,$$

so that

$$u_1^r(y) = \int_{\Delta_1 \cap C\bar{D}_r} K(x, y) d\nu(x)$$

is positive harmonic in Ω.

If we take $\omega = D_r \cap \Omega$ and put

$$u^r(y) = \int K(x, y) d\mu_1^r(x)$$

(we use μ_1^r instead of μ_1), then

$$u^r(y) \geqq u_1^r(y)$$

as in [1].

Using the fact that $u_1^r(y)$ is positive and harmonic for all positive r less than r_0, we can conclude that C_r is non-compact in Ω. The carrier of μ_1^r is contained in $\overline{C_r \cap \Omega} \cap \Delta \subset C_r \cap \Delta$. As Ikegami ([1]) did, we can prove that μ_1^r is not canonical for all positive r less than r_0. This implies that the carrier of μ_1^r contains at least one non-minimal point of Δ. If $r_1 \neq r_2$, then $C_{r_1} \cap C_{r_2} = \phi$. Consequently we can conclude that Δ_0 is uncountable if it is not void.

References

[1] T. Ikegami: On the non-minimal Martin boundary points. Nagoya Math. J., **29**, 287-290 (1967).

[2] R. S. Martin: Minimal positive harmonic functions. Trans. Amer. Math. Soc., **49**, 137-172 (1941).