81. *Moments of the Last Exit Times*

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We consider the typical stochastic processes and give the several conditions for the existence of arbitrary order moments of the last exit times from given sets. The answers will vary quite according to the dimension of the state space.

§ 1. **Stable process.** In this section we will consider a stable process, $X(t)$, on R^d (d-dimensional Euclidean space) having index α, $0<\alpha\leqq 2$, and normalized so that the path functions are right continuous with left hand limits at every point. For simplicity, we restrict our discussion to symmetric stable processes, that is, processes with stationary independent increments having continuous transition density

$$p(t, x, y)=p(t, y-x)=(2\pi)^{-d}\int_{R^d} e^{-i(\theta, y-x)}e^{-t|\theta|^\alpha}d\theta.$$

We will write P_x and E_x for the conditional probability and expectation under the condition $X(0)=x$.

For a bounded Borel (more generally, analytic) set $B\subset R^d$, let

$$\begin{aligned} T_B &= \sup\{t\geqq 0;\, X(t)\in B\} \\ &= 0, \qquad \text{if } X(t)\notin B \text{ for all } t>0, \end{aligned}$$

denote the *last exit time* of B, and let

$$\begin{aligned} V_B &= \inf\{t>0;\, X(t)\in B\} \\ &= \infty, \qquad \text{if } X(t)\notin B \text{ for all } t>0, \end{aligned}$$

be the *first hitting time* into B.

Theorem 1. *In the transient symmetric stable process, namely the case $d>\alpha$, the relation*

$$P_x[T_B\in dt]=\int_{\bar{B}} p(t, x, y)\mu_B(dy)dt \text{ [2)]}$$

holds. Here μ_B is the equilibrium (capacitary) measure of B.

This theorem is due to S. Watanabe [6]. Recently by the same principle, S. C. Port [2] gave the following result:

$$P_x[T_B>t]=\int_{\bar{B}}\int_t^\infty p(s, x, y)ds\,\mu_B(dy)$$

from which our theorem immediately follows.

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1) Financial support from the Sakkokai Foundation for the author is gratefully acknowledged.

2) The notation $\bar{B}$ means closure of B.

Theorem 2. *For the symmetric stable process, let B be a bounded Borel (or analytic) subset of R^d. Then $E_x[T_B{}^k]$ exists if and only if* $0<\alpha<\dfrac{d}{k+1}$. *Here $E_x[T_B{}^k]$ means the k-th order moment of T_B.*

Proof. First we consider the case $d>\alpha$. By means of Theorem 1, we obtain

$$E_x[T_B{}^k]=\int_0^\infty t^k\int_{\bar{B}}p(t, y-x)\mu_B(dy)dt.$$

To estimate $E_x[T_B{}^k]$, we represent it as the sum of two integrals,

$$\begin{aligned}E_x[T_B{}^k]&=I_1+I_2\\&=\left[\int_0^\beta+\int_\beta^\infty t^k\int_{\bar{B}}p(t, y-x)\mu_B(dy)dt\right],\end{aligned}$$

where β is a certain positive number.

In order to show the finiteness of the first part I_1, we recall the fact that the density $p(t, y-x)$ is bounded. Hence we have

$$\begin{aligned}I_1&=\int_0^\beta t^k\int_{\bar{B}}p(t, y-x)\mu_B(dy)dt\\&\leq K\cdot C(B)\int_0^\beta t^k dt<+\infty,\end{aligned}$$

where $C(B)$ denotes the natural capacity of B.

As for the second term I_2, applying the familiar scaling relationship

$$(1)\qquad p(t, x)=t^{-d/\alpha}p(1, xt^{-1/\alpha}),$$

we get

$$\begin{aligned}I_2&=\int_\beta^\infty t^k\int_{\bar{B}}t^{-d/\alpha}p(1, (y-x)t^{-1/\alpha})\mu_B(dy)dt\\&\leq K'\cdot C(B)\int_\beta^\infty t^{k-\frac{d}{\alpha}}dt.\end{aligned}$$

The last integral is bounded if $k-\dfrac{d}{\alpha}<-1$.

On the contrary, if we put

$$\delta=\min\{|t|;\ 2p(1, t^{-1/\alpha}(y-x))\geqq p(1, 0)\},$$

then

$$\begin{aligned}E_x[T_B{}^k]&>\int_\delta^\infty t^{k-\frac{d}{\alpha}}p(1, t^{-1/\alpha}(y-x))dt\int_{\bar{B}}\mu_B(dy)\\&\geqq\frac{p(1, 0)}{2}\cdot C(B)\int_\delta^\infty t^{k-\frac{d}{\alpha}}dt.\end{aligned}$$

Consequently the moments do not exist if $k-\dfrac{d}{\alpha}\geqq-1$; in this case $d>\alpha\geqq\dfrac{d}{k+1}$ necessarily holds. When $\alpha\geqq d$, the processes are recurrent and so the last exit times from B are infinite with probability one. This establishes the truth of our theorem.

Next we turn our attention to the first hitting time V_B of B. The ordinary average of the first hitting times to bounded set does not exist even in the case of recurrent stable process. This is derived from the recent investigations of S. C. Port [3]. He showed that

$$P_x[V_B>t] \sim L\cdot t^{-1+\frac{1}{\alpha}}, \qquad \text{as } t\to\infty,$$

where $1<\alpha<2$ and L is a constant depending x, α, and B. However, the expectations eliminating the measure of the case $V_B=\infty$ are closely related to those of the last exit time.

Theorem 3. *For the symmetric stable process, let B be a bounded set of R^d. Let $0<\alpha<\dfrac{d}{k+1}$. Then*

$$E_x[V_B{}^k;\ V_B<\infty]=\int_{\{\omega;V_B<\infty\}} V_B{}^k(\omega)P_x[d\omega]$$

is finite for all $x\in R^d$.

Proof. We have

$$\begin{aligned}E_x[V_B{}^k;\ V_B<\infty]&=\sum_{n=0}^{\infty}\int_n^{n+1} t^k d_t P_x[V_B\leqq t]\\&=O(1)\sum_{n=1}^{\infty} n^k P_x[n<V_B\leqq n+1]\\&\leqq O(1)\sum_{n=1}^{\infty} n^k\int_{R^d} p(n,\ y-x)P_y[V_B\leqq 1]dy.\end{aligned}$$

In view of the scaling relationship (1) once again, the last expression is equal to

$$O(1)\sum_{n=1}^{\infty} n^k\int_{R^d} n^{-d/\alpha}p(1,\ n^{-1/\alpha}(y-x))P_{n^{-1/\alpha}y}\left[V_{B'}\leqq\frac{1}{n}\right]dy,$$

where $B'=\{x;\ n^{1/\alpha}x\in B\}$.

Thus we get

$$\begin{aligned}E_x[V_B{}^k;\ V_B<\infty]&\leqq O(1)\sum_{n=1}^{\infty} n^{k-\frac{d}{\alpha}}P_{n^{-1/\alpha}x}\left[V_{B'}\leqq 1+\frac{1}{n}\right]\\&=O(1)\sum_{n=1}^{\infty} n^{k-\frac{d}{\alpha}}P_x[V_B\leqq n+1]\\&\leqq O(1)\sum_{n=1}^{\infty} n^{k-\frac{d}{\alpha}}.\end{aligned}$$

Therefore the moment is finite if $n-\dfrac{d}{\alpha}<-1$.

§ 2. Markov chain and random walk. In this section X_n will always be an irreducible Markov chain (or random walk) with states in a denumerable set and with n-th step transition probability matrix $P_n(x,y)$. We say that the Markov chain is *irreducible* if all states communicate with each other. As previous section, V_B, T_B are respectively the time of first entrance and last exit into a finite nonempty set B. Our paper rests heavily on the works of S. C. Port [1].

We first investigate a general criterion for irreducible Markov chain.

Theorem 4. *Suppose Markov chain is irreducible. For any finite nonempty set B, $E_x[T_B^k]$ is finite if and only if*

$$\sum_{n=1}^{\infty} n^k P_n(0, 0) < +\infty \text{ [3)]} \tag{2}$$

Proof. For $n>0$ we have

$$P_x[T_B=n] = \sum_{y\in B} P_n(x, y) P_y[V_B=\infty].$$

Hence

$$E_x[T_B^k] = \sum_{n=1}^{\infty} n^k P_x[T_B=n] = \sum_{y\in B}\sum_{n=1}^{\infty} n^k P_n(x, y) P_y[V_B=\infty].$$

By our assumption, we can find an $l(x)$ and $m(y)$ such that $P_l(x, 0)>0$ and $P_m(0, y)>0$. Then for any $n\geqq 0$,

$$P_{n+l+m}(x, y) \geqq P_l(x, 0) P_n(0, 0) P_m(0, y) = \gamma P_n(0, 0), \qquad \gamma>0.$$

Similarly for some integers $r(x)$ and $s(y)$ we get

$$P_{n+r+s}(0, 0) \geqq \gamma' P_n(x, y), \qquad \gamma'>0.$$

Using the above inequalities and noting the finiteness of set B, we obtain

$$\begin{aligned} E_x[T_B^k] &= O(1) \sum_{y\in B}\sum_{n=1}^{\infty} n^k P_n(0, 0) P_y[V_B=\infty] \\ &= O(1) \sum_{n=1}^{\infty} n^k P_n(0, 0). \end{aligned}$$

This completes the proof of our theorem.

In the following, we concerned with the random walk on the integer lattice points in d-dimensional Euclidean space. In analogy with the criterion for transience, we get the following condition in terms of the characteristic function $\phi(\theta) = \sum_x P(0, x) e^{ix\cdot\theta}$.

Theorem 5. *In any d-dimensional random walk, the series $\sum_{n=1}^{\infty} n^k P_n(0, 0) < \infty$ if and only if*

$$\lim_{t\uparrow 1} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \mathrm{Re}\left[\frac{1}{(1-t\phi(\theta))^{k+1}}\right] d\theta < \infty. \tag{3}$$

Proof. Recalling that

$$P_n(0, 0) = (2\pi)^{-d} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \phi^n(\theta) d\theta,$$

we find that, for $0 \leq t < 1$,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(n+k)!}{n!} P_n(0, 0) t^n &= (2\pi)^{-d} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} [1-t\phi(\theta)]^{-k-1} d\theta \\ &= (2\pi)^{-d} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \mathrm{Re}[1-t\phi(\theta)]^{-k-1} d\theta. \end{aligned}$$

3) Here 0 means an arbitrary point of state space. If Markov chain is recurrent, T_B are infinite with probability one and condition (2) is not satisfied. So the conclusion of theorem 4 is trivial.

But the series $\sum_{n=1}^{\infty} n^k P_n(0, 0)t^n < \infty$, if and only if

$$\sum_{n=1}^{\infty} \frac{(n+k)!}{n!} P_n(0, 0)t^n < \infty.$$

Thus

$$\sum_{n=1}^{\infty} n^k P_n(0, 0) = \lim_{t \uparrow 1} \sum_{n=1}^{\infty} n^k P_n(0, 0)t^n < \infty$$

if and only if the expression (3) is finite.

Now we show that the moments of last exit times exist in comparatively higher dimensional random walks.

Theorem 6. *Suppose the random walk is irreducible. Then if* $d > 2(k+1)$, *we always have* $E_x[T_B{}^k] < \infty$.

Proof. As our random walk is irreducible, there are at least $(d+1)$ affinely independent points x such that $P(0, x) > 0$, and that suffices to satisfy condition (3) of P 7.6 in F. Spitzer [5]. Applying that proposition, we have $P_n(0, 0) \leqq An^{-d/2}$ for some constant $A > 0$. Hence

$$\sum_{n=1}^{\infty} n^k P_n(0, 0) \leqq A \sum_{n=1}^{\infty} n^{k-d/2}.$$

Consequently if $k - \frac{d}{2} < -1$, the moment always exists.

If the partial sums S_n, which constitute the random walk, are in the domain of attraction of certain stable laws, we can obtain the result analogous to theorem 2.

Theorem 7. *Suppose the irreducible random walk has partial sums* S_n *such that* $S_n/n^{1/\alpha}L(n)$ *converges in distribution to a stable law with density* $p_\alpha(x)$, *where* $p_\alpha(0) \neq 0$. *Here* $L(\cdot)$ *is a slowly varying function. Then we have* $E_x[T_B{}^k] < \infty$ *whenever* $0 < \alpha < \frac{d}{k+1}$, *while* $E_x[T_B{}^k] = \infty$ *when* $2 > \alpha > \frac{d}{k+1}$. *For* $\alpha = \frac{d}{k+1}$, *the result depends on what* $L(n)$ *is.*

Proof. The local limit theorem for multi-dimensional lattice distribution (Th. 6.1 in E. L. Rvačeva [4]) can easily be modified to cover the case when the random walk is temporally periodic with period λ. The result is that when

$$\frac{S_n}{n^{1/\alpha}L(n)}$$

converges in distribution to a stable law with density p_α, then we obtain

$$P_{\lambda n}(0, 0) \sim \lambda p_\alpha(0)(\lambda n)^{-d/\alpha} L(\lambda n)^{-d}, \qquad n \to \infty.$$

Combining this formula and theorem 4, we can prove the conclusion of theorem 7.

Remark. We remark that the results analogous to theorem 3 can be obtain for irreducible Markov chains and random walks. It suffices to note the fact

$$P_x[V_B=n]=\sum_{y\in B}\{P_n(x,y)-P_x[V_B\leqq n-1, X_n=y]\}$$
$$\leqq\sum_{y\in B}P_n(x,y).$$

We do not discuss these in detail.

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