# 107. *On a Certain Class of Univalent Functions*

By Tetsujirô KAKEHASHI

University of Osaka Prefecture

(Comm. by Kinjirô KUNUGI, M.J.A., June 12, 1967)

Let us consider a simply connected polygon which has $2n$ sides parallel to the real axis or imaginary axis in the $w$-plane. If we call its vertices $w_1, w_2, \cdots, w_2$ and denote its interior angles $\pi\alpha_1, \pi\alpha_2, \cdots, \pi\alpha_{2n}$ respectively, $\alpha_k$ takes the value 1/2 or 3/2, and $\sum_{k=1}^{2n}\alpha_k$ is equal to $2n-2$.

We can construct the function $w=f(z)$ which maps the interior of unit circle $|z|<1$ onto the interior of this polygon by

$$\frac{dw}{dz}=K(z-z_1)^{\alpha_1-1}(Z-z_2)^{\alpha_2-1}\cdots(z-z_{2n})^{\alpha_{2n}-1}, \tag{1}$$

where $z_k=e^{i\theta_k}(0\leqq\theta_1<\theta_2<\cdots<\theta_{2n}<2\pi)$ are points on the unit circle $|z|=1$, and $k$ is a constant complex number. The equality (1) is known as Schwarz-Christoffel's formula.

If we put $z_k^{-1}=\varepsilon_k$, we have

$$\frac{dw}{dz}=C(1-\varepsilon_1 z)^{\delta_1}(1-\varepsilon_2 z)^{\delta_2}\cdots(1-\varepsilon_{2n}z)^{\delta_{2n}}, \tag{2}$$

where $C$ is a constant, $\delta_k$ is equal to 1/2 or $-1/2$ and $\sum_{k=1}^{2n}\delta_k$ is equal to $-2$. And square roots in (2) mean to take the branch such that $\sqrt{1}=1$. The function $\frac{dw}{dz}$ above defined is analytic for $|z|<1$ and $w=f(z)$ is analytic and univalent for $|z|<1$.

Next we consider a polygon shown in Fig. 1. In this case, we can write signs of $\delta_k$ in order and if we take apart suitable four minus signs, we can arrange a sequence of couples $(-+)$ or $(+-)$ as follows,

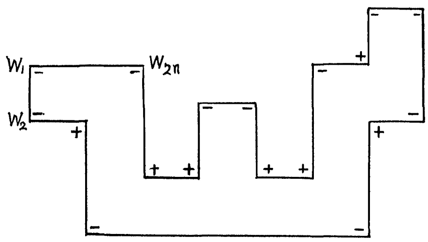


Fig. 1

$$\ominus\ominus(+-)(-+)\ominus\ominus(-+)(-+)(+-)(-+)(+-). \tag{3}$$

We shall denote a class of functions $w=f(z)$ which map the interior of unit circle respectively onto the interior of a polygon which has the nature above mentioned by the symbol $S_0$. For a function which belongs to the class $S_0$, we have the following theorem.

**Theorem.** *Let $w=f(z)$ be a function which belongs to the class $S_0$, and let*

(4) $\quad w=f(z)=A+C(z+A_2z^2+\cdots+A_nz^n+\cdots): |z|<1$
*be the Taylor's expansion of* $w=f(z)$. *Then coefficients* $A_n$ *satisfy*
(5) $$|A_n|<n : n=2, 3, \cdots.$$

In the proof of this theorem, we consider the following lemma.

**Lemma.** *Let* $\zeta_k$: $k=1, 2 \cdots, 2N$ *be points on the unit circle such that* $\zeta_k=e^{i\theta_k}(0\leqq\theta_1\leqq\theta_2\leqq\cdots\leqq\theta_{2N}\leqq 2\pi)$, *and* $G(z)$ *be a function represented by*

$$G(z)=\frac{z-\zeta_2}{z-\zeta_1}\,\frac{z-\zeta_4}{z-\zeta_3}\cdots\frac{z-\zeta_{2N}}{z-\zeta_{2N-1}}.$$

*Then, for* $|z|<1$, *the function* $G(z)$ *takes values on a half plane bordered by a line which passes the origin.*

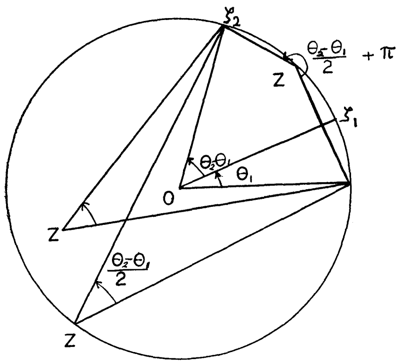

Fig. 2

**Proof.** In Fig. 2, when $|z|=1$, we have

$$\arg\frac{z-\zeta_2}{z-\zeta_1}=\begin{cases}\frac{1}{2}(\theta_2-\theta_1): & z\,\overline{\in}\,\overrightarrow{\zeta_1\zeta_2}\\ \frac{1}{2}(\theta_2-\theta_1)+\pi: & z\in\overrightarrow{\zeta_1\zeta_2}\end{cases}$$

and when $|z|<1$, we have

$$\frac{1}{2}(\theta_2-\theta_1)<\arg\frac{z-\zeta_2}{z-\zeta_1}<\frac{1}{2}(\theta_2-\theta_1)+\pi.$$

Accordingly, when $z$ varies on the unit circle, if $z$ is not on any one of arcs $\overrightarrow{\zeta_1\zeta_2}, \overrightarrow{\zeta_3\zeta_4}\cdots\overrightarrow{\zeta_{2N-1}\zeta_{2N}}$, $\arg G(z)$ is equal to

$$\Theta=\frac{1}{2}(-\theta_1+\theta_2-\theta_3+\theta_4-\cdots-\theta_{2N-1}+\theta_{2N}),$$

and if $z$ is on any one of these arcs, $\arg G(z)$ is equal to $\Theta+\pi$. And when $z$ is an interior point to the unit circle, we have $\Theta<\arg G(z)<\Theta+2\pi$. Thus the lemma has been proved.

Now we shall prove the theorem. When a function $w=f(z)$ belongs to the class $S_0$, $\frac{dw}{dz}$ can be written from (2) as follows,

(6) $$\frac{dw}{dz}=C\prod_{k=1}^{4}(1-\varepsilon_{1k}z)^{-1/2}\left[\prod_{\mu}\frac{1-\varepsilon_{2,2\mu}z}{1-\varepsilon_{2,2\mu-1}z}\right]^{1/2}\left[\prod_{\nu}\frac{1-\varepsilon_{3,2\nu}z}{1-\varepsilon_{3,2\nu-1}z}\right]^{-1/2},$$

where $z_{1k}=\varepsilon_{1k}^{-1}$ are points correspond to four minus signs removed suitably in (3), $(z_{2,2\mu-1}=\varepsilon_{2,2\mu-1}^{-1}, z_{2,2\mu}=\varepsilon_{2,2\mu}^{-1})$ are couples correspond to $(-+)$, and $(z_{3,2\nu-1}=\varepsilon_{3,2\nu-1}^{-1}, z_{3,2\nu}=\varepsilon_{3,2\nu}^{-1})$ are couples correspond to $(+-)$ in (3).

We can verify that the Taylor's expansion of $\prod_{k=1}^{4}(1-\varepsilon_{1k}z)^{-1/2}$ is majorated by $(1-z)^{-2}=1+2z+3z^3+\cdots+nz^{n-1}+\cdots$, because $(1-z)^{-1/2}=1+\frac{1}{2}z+\frac{3}{8}z^2+\cdots$ is a power series with positive coefficients. That is, if we put

$$\prod_{k=1}^{4}(1-\varepsilon_{1k}z)^{-1/2}=1+\alpha_1 z+\alpha_2 z^2+\cdots+\alpha_n z^n+\cdots,$$

we have $|\alpha_{n-1}|\leqq n$ and the equality is valid only when all $z_{1k}$ coincide with one point.

For $|z|<1$, functions $\prod_{\mu}\frac{1-\varepsilon_{2,2\mu}z}{1-\varepsilon_{2,2\mu-1}z}$ and $\prod_{\mu}\frac{1-\varepsilon_{3,2\nu}z}{1-\varepsilon_{3,2\nu-1}z}$ in (4) take values respectively on a half plane defined in the lemma. If we define that square roots take respectively the branch such that $\sqrt{1}=1$, $\left[\prod_{\mu}\frac{1-\varepsilon_{2,2\mu}z}{1-\varepsilon_{2,2\mu-1}z}\right]^{1/2}$, and $\left[\prod_{\nu}\frac{1-\varepsilon_{3,2\nu}z}{1-\varepsilon_{3,2\nu-1}z}\right]^{-1/2}$ take values respectively on a quarter plane bordered by two lines meet at right angle in the origin. Accordingly, for $|z|<1$, the function $\left[\prod_{\mu}\frac{1-\varepsilon_{2,2\mu}z}{1-\varepsilon_{2,2\mu-1}z}\right]^{1/2}$ $\left[\prod_{\nu}\frac{1-\varepsilon_{3,2\nu}z}{1-\varepsilon_{3,2\nu-1}z}\right]^{-1/2}$ takes values on a half plane bordered by a line which passes the origin.

As the half plane contains the unit in its interior, the product of this function and $e^{i\varphi}\left(-\frac{\pi}{2}<\varphi<\frac{\pi}{2}\right)$ takes values which have positive real parts for $|z|<1$. If we write the Taylor's expansion of this function as follows,

$$(7)\qquad \left[\prod_{\mu}\frac{1-\varepsilon_{2,2\mu}z}{1-\varepsilon_{2,2\mu-1}z}\right]^{1/2}\left[\prod_{\nu}\frac{1-\varepsilon_{3,2\nu}z}{1-\varepsilon_{3,2\nu-1}z}\right]^{-1/2}=1+\beta_1 z+\beta_2 z^2+\cdots+\beta_n z^n+\cdots,$$

it is known that inequalities $|e^{i\varphi}\beta_n|\leqq 2\cos\varphi\leqq 2$ follow, that is, we have $|\beta_n|\leqq 2$. Now we can verify that the Taylor's expansion (7) is majorated by $\frac{1+z}{1-z}=1+2z+2z^2+\cdots+2z^n+\cdots$.

Accordingly, the Taylor's expansion

$$\frac{1}{C}\frac{dw}{dz}=1+a_1 z+a_2 z^2+\cdots+a_n z^n+\cdots:|z|<1$$

can be majorated by

$$\frac{1}{(1-z)^2}\frac{1+z}{1-z}=1+2^2 z+3^2 z^2+\cdots+n^2 z^{n-1}+\cdots,$$

and we have $|a_{n-1}|<n^2$. $|A_n|=\frac{|a_{n-1}|}{n}<n$ follows at once. Thus the theorem has been established.

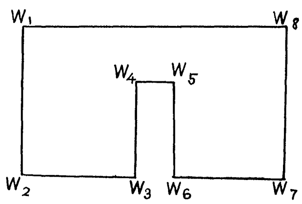

Fig. 3

**Remark.** The equality $|A_n|=n$ can be satisfied only when $z_1=z_2=z_3=z_6=z_7=z_8=\varepsilon$, $z_4=z_5=-\varepsilon$ $(|\varepsilon|=1)$ as the limit case of a polygon in Fig. 3.