127. *Notes on Hilbert Transforms of Vector Valued Functions in the Complex Plane and Their Boundary Values*

By Kiyoshi ASANO

Research Institute for Mathematical Sciences, Kyoto University, Kyoto

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1. Introduction. Let $\mathfrak{H}$ be a complex Hilbert space with norm $\|\ \|$ and ρ be a non-negative measure on $R=(-\infty,\infty)$ which is finite on every bounded Borel set. We denote by $L_\rho(R;\mathfrak{H})=L_\rho$ the set of all $\mathfrak{H}$-valued strongly ρ-measurable functions $f(t)$ on R satisfying the condition $\int_{-\infty}^{\infty}\|f(t)\|\,d\rho(t)<\infty$. For each $f(t)\in L_\rho$ we define its real and complex Hilbert transforms $H_\rho[f]$ and $\mathscr{H}_\rho[f]$ with respect to ρ as follows (if the right members exist):

$$(1)\qquad H_\rho[f](x)=\text{p.v.}\int_{-\infty}^{\infty}\frac{1}{t-x}f(t)d\rho(t)$$

$$=\text{s}-\lim_{\varepsilon\downarrow 0}\int_{|t-x|>\varepsilon}\frac{1}{t-x}f(t)d\rho(t)\qquad(x\in R),$$

$$(2)\qquad \mathscr{H}_\rho[f](z)=\int_{-\infty}^{\infty}\frac{1}{t-z}f(t)d\rho(t)\qquad(\text{im}\,z\neq 0),$$

where the integrals are taken in the sense of Bochner and s$-$lim means the limit in the strong topology of $\mathfrak{H}$. Clearly $\mathscr{H}_\rho[f](z)$ exists for all $z(\text{im}\,z\neq 0)$ and it is a $\mathfrak{H}$-valued analytic function in the upper and lower half-planes. We are concerned with the behavior of $\mathscr{H}_\rho[f](z)$ as z approaches to R from upper or lower half-plane. At first we note some measure theoretic points. Since $f(t)\in L_\rho$ is a strong limit of a sequence of step functions at $\rho-$a.e. (almost every) $t\in R$, there exists a (closed) separable subspace $\mathfrak{H}_f$ of $\mathfrak{H}$ such that $f(t)\in\mathfrak{H}_f$ for $\rho-$a.e. $t\in R$. Therefore we may assume without loss of generality that $\mathfrak{H}$ itself is separable. We put $\rho=\rho_0+\rho_1$ where ρ_0 is the singular part of ρ and ρ_1 is the absolutely continuous part of ρ (with density $\rho'(t)$). Then $f(t)\rho'(t)$ is strongly Lebesgue measurable, and by the standard argument we can see that for a.e. $t\in R$ $\int_{t-h}^{t+h}\|f(s)\rho'(s)-f(t)\rho'(t)\|\,ds=o(h)$ and $\int_{t-h}^{t+h}\|f(s)\|\,d\rho_0(s)=o(h)$ as $h\downarrow 0$. The set of all such t is denoted by $\Lambda_{\rho,f}$. Clearly $(\Lambda_{\rho,f})^c$ is a null set ($=$a set of Lebesgue measure 0) and $\text{s}-\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}f(t)d\rho(t)$ $=f(t)\rho'(t)$ for each $t\in\Lambda_{\rho,f}$.

Now our result reads as follows:

Theorem 1. *Let $f(t) \in L_\rho(R; \mathfrak{H})$. Then*

(i) *$H_\rho[f](x)$ exists for a.e. $x \in R$,*

(ii) $\text{s}-\lim_{\varepsilon\downarrow 0} \mathcal{H}_\rho[f](x \pm i\varepsilon) \equiv \mathcal{H}_\rho[f](x \pm i0)$ *exists whenever $H_\rho[f](x)$ exists and $x \in \Lambda_{\rho,f}$. For such x there holds*

$$\mathcal{H}_\rho[f](x \pm i0) = H_\rho[f](x) \pm \pi i f(x)\rho'(x), \tag{3}$$

(iii) *in particular for a.e. $x \in R$, $\mathcal{H}_\rho[f](x \pm i0)$ exists and* (3) *holds.*

We can rewrite the above theorem in a slightly different form. Let $V(R; \mathfrak{H}) = V$ be the set of all $\mathfrak{H}$-valued functions on R which are of strongly bounded variation. We can also define the real and complex Hilbert transforms of $v(t) \in V$ as follows (if the right members exist):

$$H[v](x) = \text{p.v.}\int_{-\infty}^{\infty} \frac{1}{t-x} dv(t), \tag{4}$$

$$\mathcal{H}[v](z) = \int_{-\infty}^{\infty} \frac{1}{t-z} dv(t) \qquad (\text{im } z \neq 0), \tag{5}$$

where the integrals are taken in the sense of Riemann-Stieltjes integral. $\mathcal{H}[v](z)$ is a $\mathfrak{H}$-valued analytic function in the upper and lower half-planes. Concerned with the behavior of $\mathcal{H}[v](z)$ near the real line, we may also assume that $\mathfrak{H}$ is separable, since $v(t) \in V$ is separably-valued. We put $\sigma(t) =$ *total variation of* $v(t)$ in $(-\infty, t)$. Then $\sigma(t)$ is a non-decreasing function on R with finite variation $\sigma(\infty)$ and $v(t)$ is strongly absolutely continuous with respect to $\sigma(t)$ (in fact $\| v(t) - v(s) \| \leq \sigma(t) - \sigma(s)$ for any $s \leq t$). Hence by the theorem of Gelfand [4] or Dunford-Pettis [3], there exists a $\mathfrak{H}$-valued strongly σ-measurable function $g(t)$ on R such that $\| g(t) \| = 1$ and $v(t) - v(a) = \int_a^t g(s) d\sigma(s)$ for $\sigma -$a.e. $t \in R$. Since the integral is taken in the sense of Bochner, $v(t)$ is strongly differentiable at a.e. $t \in R$ and $v'(t) = g(t)\sigma'(t)$. If $t \in \Lambda_v = \Lambda_{\sigma,g}$, the total variation $\omega_t(h)$ of $w_t(h) = v(t+h) - v(t) - v'(t)h$ in $(-h, h)$ is $o(h)$ as $h \downarrow 0$. Clearly $\int_{-\infty}^{\infty} \| g(s) \| d\sigma(s) = \sigma(\infty) < \infty$ and $g \in L_\sigma(R; \mathfrak{H})$. There also hold $\mathcal{H}[v](z) = \mathcal{H}_\sigma[g](z)$ and $H[v](x) = H_\sigma[g](x)$. Therefore Theorem 1 implies

Theorem 2. *Let $v(t) \in V(R; \mathfrak{H})$. Then*

(i) *$H[v](x)$ exists for a.e. $x \in R$,*

(ii) $\text{s-}\lim_{\varepsilon\downarrow 0} \mathcal{H}[v](x \pm i\varepsilon) \equiv \mathcal{H}[v](x \pm i0)$ *exists whenever $H[v](x)$ exists and $x \in \Lambda_v$. For such x there holds*

$$\mathcal{H}[v](x \pm i0) = H[v](x) \pm \pi i v'(x), \tag{6}$$

(iii) *in paticular for a.e. $x \in R$, $\mathcal{H}[v](x \pm i0)$ exists and* (6) *holds.*

We will give the proof of Theorem 1 in the next section. The method is quite the same as the elegant method due to Calderón and Zygmund [2]. Finally we give some comments on the problem. The result stated above has been known for many years in the scalar case and may be known in the vector case, but no straightforward proof has appeared in the litterature. On the other hand, it can not be generalized to the Banach space (even if we assume reflexivity) since the result is closely connected with the Fourier transform method (cf. [6]).

2. Proof of Theorem 1. Let I be any finite open interval. To prove (i), it is sufficient to show that $H_\rho[f](x)$ exists for a.e. $x\in I$. With no loss of generality we may assume that $f(t)\equiv 0$ for $t\notin I$. Let $\mathfrak{B}$ be a σ-field of Borel sets in R. We denote by $|A|$ the Lebesque measure of $A\in\mathfrak{B}$, and by $I(x,r)$ the open interval $(x-r, x+r)$. We put for $A\in\mathfrak{B}$ $\sigma(A)=\int_A \|f(t)\|d\rho(t)$ and $v(A)=\int_A f(t)d\rho(t)$. Obviously σ is a non-negative finite measure on R and v is a $\mathfrak{H}$-valued measure (the integral is taken in the sense of Bochner). Clearly $\|v(A)\|\le\sigma(A)$.

Fix α, $0<\alpha<1$. Let $A\in\mathfrak{B}$ be any set such that $A\subset I(x,r)$ and $|A|\ge\alpha|I(x,r)|$. Then $\lim_{r\downarrow 0}\sigma(A)/|A|=\sigma'(x)=\|f(x)\|\rho'(x)$ and $\kappa(x)\equiv\sup_{r>0}\sigma(A)/|A|<\infty$ for a.e. $x\in I$. Hereafter we fix α so that $0<\alpha<1/2$. We denote by e_σ (resp. e_ρ) a support of the singular part of σ (resp. ρ). We may assume that $e_\sigma\subset e_\rho$.

Now it is sufficient to prove the following:

For any $\eta>0$ there exists an $A_1\in\mathfrak{B}$ such that

(i) $|A_1|<\eta$,

(ii) $H_\rho[f](x)$ *exists for a.e.* $x\in I\cap A_1^c$.

For any $y>0$ we put $E_y=\{x\in I; y<\kappa(x)\le\infty\}$. Suppose α, $0<\alpha<1/2$, is given. Then there exists $y>0$ such that $|E_y|<\eta/2$. Let A be an open set such that i) $E_y\subset A\subset I$, ii) $e_\rho\cap I\subset A$, and iii) $|A|<\eta/2$.

We consider the non-negative measure $\nu(E)=\int_{A\cap E}dt$, $E\in\mathfrak{B}$. Clearly we have

$$\lim_{r\downarrow 0}\frac{\nu(I(x,r))}{|I(x,r)|}=\lim_{r\downarrow 0}\frac{1}{2r}\int_{A\cap I(x,r)}dt=0 \qquad \textit{for a.e. } x\in A^c.$$

As an open set A is written as a disjoint sum of at most denumerable number of open intervals: $A=\Sigma I_k$. Let t_k be the center of I_k. We put $A_1=\bigcup I_k'$ where $I_k'=(t_k-|I_k|, t_k+|I_k|)$. Then A_1 is an open set and $|A_1|<\eta$.

Take any I_k. At least one of its end points, say a, is in I, and $a\notin A$ implies $a\notin E_y$. Hence $\kappa(a)\le y$. Since $\alpha<1/2$, we have $\sigma(I_k)\le y|I_k|$ for any k. Furthermore because of ii) σ is absolutely continuous on $I\cap A^c$ and because of i) $0\le\sigma'(x)\le y$ for a.e. $x\in I\cap A^c$.

Put

$$h(t)=\begin{cases} f(t)\rho'(t) & t\in I\cap A^c, \\ \dfrac{1}{|I_k|}\displaystyle\int_{I_k} f(t)d\rho(t)=\dfrac{1}{|I_k|}v(I_k) & t\in I_k, \\ 0 & t\notin I. \end{cases}$$

Then we have $\|h(t)\|\le y$ for $t\in I$.

After these observations we consider the behavior of

$$H_\rho^\varepsilon[f](x)=\int_{|t-x|>\varepsilon}\frac{1}{t-x}f(t)d\rho(t) \qquad (\varepsilon>0)$$

for $x\in I\cap A_1^c$. Obviously

$$\begin{aligned} H_\rho^\varepsilon[f](x)&=\int_{|t-x|>\varepsilon}\frac{1}{t-x}h(t)dt \\ &\quad+\sum_{k=1}^{\infty}\int_{I_k\cap(|t-x|>\varepsilon)}\frac{1}{t-x}\left\{f(t)d\rho(t)-\frac{1}{|I_k|}v(I_k)dt\right\} \\ &=J_1(x,\varepsilon)+J_2(x,\varepsilon)+J_3(x,\varepsilon), \end{aligned}$$

where

$$J_1(x,\varepsilon)=\int_{|t-x|>\varepsilon}\frac{1}{t-x}h(t)dt,$$

$$J_2(x,\varepsilon)=\Sigma_{I_k\cap I(x,\varepsilon)\neq\phi}\int\cdots=\Sigma_1\int\cdots,$$

$$J_3(x,\varepsilon)=\Sigma_{I_k\cap I(x,\varepsilon)=\phi}\int\cdots=\Sigma_2\int\cdots.$$

Since $h(t)\in L^2(R;\mathfrak{H})$, the application of classical method of Hilbert transform (see [6]) (combined with the lemma in this section) will yield the existence of $\text{s-}\lim_{\varepsilon\downarrow 0} J_1(x,\varepsilon)=J_1(x,0)$ at a.e. $x\in R$.

Next we consider J_2. Since $x\in I\cap A_1^c$ (hence $x\notin I_k'$) and $I_k\cap I(x,\varepsilon)\neq\phi$ in Σ_1, we have $(1/2)|I_k|<\varepsilon$ and $|t_k-x|<(1/2)|I_k|+\varepsilon$. Hence for any $t\in I_k$ $|t-x|\le|t-t_k|+|t_k-x|<3\varepsilon$, i.e. $I_k\subset I(x,3)$. Therefore we have

$$\begin{aligned} \|J_2(x,\varepsilon)\|&\le\frac{1}{\varepsilon}\Sigma_1\{\sigma(I_k)+\|v(I_k)\|\} \\ &\le\frac{1}{\varepsilon}\Sigma_1 2y\,|I_k| \\ &\le\frac{2y}{\varepsilon}\int_{A\cap I(x,3\varepsilon)}dt\to 0 \qquad \textit{for a.e. } x\in A_1^c, \text{ as } \varepsilon\downarrow 0. \end{aligned}$$

Finally we consider J_3. Put

$$a_k(x,\varepsilon)=\begin{cases}0 & \textit{if} \quad I_k\cap I(x,\varepsilon)\neq\phi, \\ 1 & \textit{if} \quad I_k\cap I(x,\varepsilon)=\phi.\end{cases}$$

$a_k(x,\varepsilon)$ is non-decreasing in $\varepsilon>0$ for fixed k and x. We have for $x\in I\cap A_1^c$

$$J_3(x,\varepsilon)=\sum_{k=1}^{\infty}a_k(x,\varepsilon)g_k(x)$$

where

$$g_k(x)=\int_{I_k}\frac{1}{t-x}\Big\{f(t)d\rho(t)-\frac{1}{|I_k|}v(I_k)dt\Big\}$$
$$=\int_{I_k}\Big\{\frac{1}{t-x}-\frac{1}{t_k-x}\Big\}\Big\{f(t)d\rho(t)-\frac{1}{|I_k|}v(I_k)dt\Big\}.$$

Since

$$\left|\frac{1}{t-x}-\frac{1}{t_k-x}\right|\le\frac{|t-t_k|}{|t-x||t_k-x|}\le\frac{|I_k|}{|t_k-x|^2}\text{[1]},$$

we have

$$\|g_k(x)\|\le\frac{|I_k|}{|t_k-x|^2}\{\sigma(I_k)+\|v(I_k)\|\}$$
$$\le\frac{2|I_k|}{|t_k-x|^2}\sigma(I_k)\equiv\zeta_k(x).$$

Hence the series $J_3(x,\varepsilon)=\Sigma a_k(x,\varepsilon)g_k(x)$ is majorized term by term by the series $\Sigma\zeta_k(x)$. Thus, if we establish the convergence of $\Sigma\zeta_k(x)$ for a.e. $x\in I\cap A_1^c$, then the Lebesgue theorem for $l^1(\mathfrak{H})$ will yield the existence of $J_3(x,\varepsilon)$ and also the existence of $\text{s-}\lim_{\varepsilon\downarrow 0}J_3(x,\varepsilon)\equiv J_3(x,0)$ for a.e. $x\in I\cap A_1^c$. However, since

$$\int_{A_1^c}\zeta_k(x)dx\le\int_{|t_k-x|>|I_k|}\frac{2|I_k|}{|t_k-x|^2}\sigma(I_k)dx$$
$$=4\Big[-\frac{1}{x}\Big]_{|I_k|}^{\infty}|I_k|\sigma(I_k)=4\sigma(I_k),$$

we have

$$\int_{A_1^c}\Big\{\sum_{k=1}^{\infty}\zeta_k(x)\Big\}dx=\sum_{k=1}^{\infty}\int_{A_1^c}\zeta_k(x)dx\text{[2]}$$
$$\le\sum_{k=1}^{\infty}4\sigma(I_k)\le\sum_{k=1}^{\infty}4y|I_k|\le 4y|A|.$$

This implies the convergence of $\Sigma\zeta_k(x)$ for a.e. $x\in I\cap A_1^c$. Thus $\text{s-}\lim_{\varepsilon\downarrow 0}H_\rho^\varepsilon[f](x)=H_\rho[f](x)$ exists for a.e. $x\in I\cap A_1^c$ and the proof of (i) is completed.

The following Lemma will complete the proof of (ii):

Lemma. *Let* $(1+|t|)^{-1}f(t)\in L_\rho(R;\mathfrak{H})$ *and* $x\in\Lambda_{\rho,f}$. *Then*

(i) $\displaystyle \text{s}-\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\frac{\varepsilon}{(t-x)^2+\varepsilon^2}f(t)d\rho(t)=\pi f(x)\rho'(x),$

(ii) $\displaystyle \text{s}-\lim_{\varepsilon\downarrow 0}\Big\{\int_{-\infty}^{\infty}\frac{t-x}{(t-x)^2+\varepsilon^2}f(t)d\rho(t)-H_\rho^\varepsilon[f](x)\Big\}=0.$

The proof of this lemma is quite similar to that in the classical case (see [6]).

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1) If $t\in I_k$ and $x\in A_1^c$ (hence $x\notin I_k'$), then $|t_k-x|\ge|I_k|\ge 2|t-t_k|$. Hence $|t-x|\ge|t_k-x|-|t-t_k|\ge\frac{1}{2}|t_k-x|$.

2) Here we have applied Lebesgue's monotone convergence theorem.

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