165. On Paracompactness and Metrizability of Spaces

By Yûkiti KATUTA
Ehime University

(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1967)

1. Introduction. In the previous note [3], we have introduced the notion of an order locally finite collection of subsets of a topological space. This is defined as follows. A collection $\{A_\lambda \mid \lambda \in \Lambda\}$ of subsets of a topological space is called *order locally finite*, if we can introduce a total order $<$ in the index set Λ such that for each $\lambda \in \Lambda$ $\{A_\mu \mid \mu < \lambda\}$ is locally finite at each point of A_λ. It is obvious that every σ-locally finite collection is order locally finite.[1)]

The purpose of this note is prove the following theorems.

Theorem 1. *Let X be a regular space. If there is an order locally finite open covering $\{G_\lambda \mid \lambda \in \Lambda\}$ of X such that for each λ the closure $\bar{G}_\lambda$ of G_λ is paracompact, then X is paracompact.*[2)]

Theorem 2. *Let X be a regular space. If there is an order locally finite open covering $\{G_\lambda \mid \lambda \in \Lambda\}$ of X such that for each λ the boundary $\mathfrak{B}(G_\lambda)$ of G_λ is compact and G_λ (more generally, every closed subset of X contained in G_λ) is paracompact, then X is paracompact.*

Theorem 3. *Let X be a collectionwise normal T_1-space. If there is an order locally finite open covering $\{G_\lambda \mid \lambda \in \Lambda\}$ of X such that for each λ the boundary $\mathfrak{B}(G_\lambda)$ of G_λ is paracompact and G_λ (more generally, every closed subset of X contained in G_λ) is paracompact, then X is paracompact.*

Theorem 4. *Let X be a collectionwise normal T_1-space. If there are a closed covering $\{F_\lambda \mid \lambda \in \Lambda\}$ and an order locally finite open covering $\{G_\lambda \mid \lambda \in \Lambda\}$ of X such that for each λ $F_\lambda \subset G_\lambda$ and F_λ is paracompact, then X is paracompact.*

Applying the metrization theorem of J. Nagata [6] and Yu. M. Smirnov [7] that a space which is the union of a locally finite collection of closed metrizable subsets is metrizable, from Theorems 1, 2, and 3 we obtain immediately the following Theorems 5, 6, and 7 respectively.

Theorem 5. *Let X be a regular space. If there is an order*

---

1) H. Tamano [9] has introduced the notion of *linearly locally finite* collections. By definition, every σ-locally finite collection is linearly locally finite and every linearly locally finite collection is order locally finite (but not conversely).

2) This theorem has been proved by Tamano [9] in the case when X is a completely regular T_1-space and $\{G_\lambda \mid \lambda \in \Lambda\}$ is lineary locally finite.

locally finite open covering $\{G_\lambda \mid \lambda \in \Lambda\}$ *of* X *such that for each* λ *the closure* $\bar{G}_\lambda$ *of* G_λ *is metrizable, then* X *is metrizable.*

Theorem 6. *Let* X *be a regular space. If there is an order locally finite open covering* $\{G_\lambda \mid \lambda \in \Lambda\}$ *of* X *such that for each* λ *the boundary* $\mathfrak{B}(G_\lambda)$ *of* G_λ *is compact and* G_λ *is metrizable, then* X *is metrizable.*

Theorem 7. *Let* X *be a collectionwise normal* T_1*-space. If there is an order locally finite open covering* $\{G_\lambda \mid \lambda \in \Lambda\}$ *of* X *such that for each* λ *the boundary* $\mathfrak{B}(G_\lambda)$ *of* G_λ *is paracompact and* G_λ *is metrizable, then* X *is metrizable.*

Theorems 6 and 7 are generalizations of A. H. Stone's theorem [8, Theorem 3] and S. Hanai's theorem [2, Theorem 7], respectively.

2. Lemmas. Lemma 1. *Let* $\{A_\lambda \mid \lambda \in \Lambda\}$ *be an order locally finite collection of subsets of a topological space* X, *and let* $\{B_\xi \mid \xi \in \Xi_\lambda\}$ *be a collection of subsets of* A_λ *which is locally finite in* X *for each* $\lambda \in \Lambda$. *Then the collection* $\{B_\xi \mid \xi \in \Xi_\lambda, \lambda \in \Lambda\}$ *is order locally finite.*

Lemma 2. *A regular space* X *is paracompact if and only if any open covering of* X *has an order locally finite open refinement.*

Lemmas 1 and 2 have been proved in [3].

Lemma 3. *Let* X *be a regular space and let* X *be the union of two subsets* A *and* B. *If* A *is compact and* B (*more generally, every closed subset of* X *contained in* B) *is paracompact, then* X *is paracompact.*

Proof. Let $\mathfrak{U}=\{U_\gamma \mid \gamma \in \Gamma\}$ be an arbitrary open covering of X. By E. Michael [4, Lemma 1], we need only prove that $\mathfrak{U}$ has a locally finite refinement. Since A is compact, it is covered by finitely many U_γ; let these be $U_1, \cdots, U_n$. Put $F=X-(U_1 \cup \cdots \cup U_n)$, then F is a closed subset of X contained in B and hence F is paracompact. Therefore the open covering $\{F \cap U_\gamma \mid \gamma \in \Gamma\}$ of F has a locally finite refinement $\mathfrak{B}$. Since F is closed in X, $\mathfrak{B}$ is locally finite in X. Thus the collection $\{U_1, \cdots, U_n\} \cup \mathfrak{B}$ is a locally finite refinement of $\mathfrak{U}$. This completes the proof.

Lemma 4. *Let* X *be a collectionwise normal space and let* X *be the union of two subsets* A *and* B. *If* A *is a paracompact closed subset and* B (*more generally, every closed subset of* X *contained in* B) *is paracompact, then* X *is paracompact.*[3]

Proof. Let $\mathfrak{U}=\{U_\gamma \mid \gamma \in \Gamma\}$ be an arbitrary open covering of X. Since A is paracompact, the open covering $\{A \cap U_\gamma \mid \gamma \in \Gamma\}$ of A has a locally finite open refinement $\{V_\delta \mid \delta \in \Delta\}$. By C. H. Dowker [1, Lemma 1], there exists a locally finite open covering $\{W_\delta \mid \delta \in \Delta\}$ of

3) This lemma has been stated by K. Morita [5, Lemma 1].

X such that $A \cap W_\delta \subset V_\delta$ for each δ. Corresponding to each $\delta \in \Delta$ we choose $\gamma(\delta) \in \Gamma$ such that $V_\delta \subset A \cap U_{\gamma(\delta)}$, and we put $S_\delta = W_\delta \cap U_{\gamma(\delta)}$. Obviously, $\mathfrak{S} = \{S_\delta \mid \delta \in \Delta\}$ is a locally finite open collection which covers A. Put $A' = X - \cup \{S_\delta \mid \delta \in \Delta\}$, then A' is a closed subset of X contained in B and hence A' is paracompact. Similarly, we obtain a locally finite open collection $\mathfrak{S}'$ such that $\mathfrak{S}'$ covers A' and each element of $\mathfrak{S}'$ is a subset of some element of $\mathfrak{U}$. Thus the collection $\mathfrak{S} \cup \mathfrak{S}'$ is a locally finite open refinement of $\mathfrak{U}$. This completes the proof.

3. **Proof of Theorem 1.** Let $\mathfrak{U} = \{U_\gamma \mid \delta \in \Gamma\}$ be an arbitrary open covering of X. Since $\bar{G}_\lambda$ is paracompact for each λ, the open covering $\{\bar{G}_\lambda \cap U_\gamma \mid \gamma \in \Gamma\}$ of $\bar{G}_\lambda$ has a locally finite open refinement $\{V_\xi \mid \xi \in \Xi_\lambda\}$. Since $\bar{G}_\lambda$ is closed in X, it is locally finite in X. Put $W_\xi = V_\xi \cap G_\lambda$ for $\xi \in \Xi_\lambda$. Of course, $\{W_\xi \mid \xi \in \Xi_\lambda\}$ is locally finite in X. Therefore by Lemma 1 the collection $\{W_\xi \mid \xi \in \Xi_\lambda, \lambda \in \Lambda\}$ is order locally finite. It is obvious that it is a refinement of $\mathfrak{U}$. Since V_ξ is open in $\bar{G}_\lambda$ and G_λ is open in X, W_ξ is open in X for each $\xi \in \Xi_\lambda, \lambda \in \Lambda$. Thus, by Lemma 2, the proof is completed.

4. **Proofs of Theorems 2 and 3.** Theorem 2 is an immediate consequence of Theorem 1 and Lemma 3, and Theorem 3 is an immediate consequence of Theorem 1 and Lemma 4.

5. **Proof of Theorem 4.** Let $\mathfrak{U} = \{U_\gamma \mid \gamma \in \Gamma\}$ be an arbitrary open covering of X. Since F_λ is paracompact for each λ, the open covering $\{F_\lambda \cap U_\gamma \mid \gamma \in \Gamma\}$ of F_λ has a locally finite open refinement $\{V_\xi \mid \xi \in \Xi_\lambda\}$. By Dowker [1, Lemma 1], there exists a locally finite open covering $\{W_\xi \mid \xi \in \Xi_\lambda\}$ of X such that $F_\lambda \cap W_\xi \subset V_\xi$ for each $\xi \in \Xi_\lambda$. Corresponding to each $\xi \in \Xi_\lambda$ we choose $\gamma(\xi) \in \Gamma$ such that $V_\xi \subset F_\lambda \cap U_{\gamma(\xi)}$, and we put $S_\xi = G_\lambda \cap W_\xi \cap U_{\gamma(\xi)}$. Then each S_ξ is open in X and $\{S_\xi \mid \xi \in \Xi_\lambda\}$ is locally finite in X. Since $\{G_\lambda \mid \lambda \in \Lambda\}$ is order locally finite, by Lemma 1 the collection $\{S_\xi \mid \xi \in \Xi_\lambda, \lambda \in \Lambda\}$ is order locally finite.

Now for each $\xi \in \Xi_\lambda$

$$S_\xi = G_\lambda \cap W_\xi \cap U_{\gamma(\xi)} \supset F_\lambda \cap W_\xi \cap U_{\gamma(\xi)}$$
$$= (F_\lambda \cap W_\xi) \cap (F_\lambda \cap U_{\gamma(\xi)}) \supset (F_\lambda \cap W_\xi) \cap V_\xi = F_\lambda \cap W_\xi.$$

Since for each $\lambda \in \Lambda$ $\{W_\xi \mid \xi \in \Xi_\lambda\}$ is a covering of X and $\{F_\lambda \mid \lambda \in \Lambda\}$ is also a covering of X, $\{S_\xi \mid \xi \in \Xi_\lambda, \lambda \in \Lambda\}$ is a covering of X. It is obvious that it refines $\mathfrak{U}$. Thus, by Lemma 2, the proof is completed.

Remark. From Theorem 4, we obtain the following:

Let X be a collectionwise normal T_1-space. If there is a σ-locally finite closed covering $\{F_\lambda \mid \lambda \in \Lambda\}$ of X such that each F_λ is paracompact, then X is paracompact.

In this result, "σ-locally finite" cannot be, however, replaced by "order locally finite". In fact, let X be the space of all ordinal numbers less than the first uncountable ordinal number with the usual topology; then the collection of all subsets of X, each of which consists of one point, is an order locally finite closed covering of X. As is well known, X is a collectionwise normal T_1-space but X is not paracompact.

References

[1] C. H. Dowker: On a theorem of Hanner. Ark. Mat., **2**, 307-313 (1952).
[2] S. Hanai: Open mappings and metrization theorems. Proc. Japan Acad., **39**, 450-454 (1963).
[3] Y. Katuta: A theorem on paracompactness of product spaces. Proc. Japan Acad., **43**, 615-618 (1967).
[4] E. Michael: A note on paracompact spaces. Proc. Amer. Math. Soc., **4**, 831-838 (1953).
[5] K. Morita: On closed mappings. Proc. Japan Acad., **32**, 539-543 (1956).
[6] J. Nagata: On a necessary and sufficient condition of metrizability. J. Inst. Polytech. Osaka City Univ., **1**, 93-100 (1950).
[7] Yu. M. Smirnov: On metrization of topological spaces. Uspehi Mat. Nauk, **6**, 100-111 (1951).
[8] A. H. Stone: Metrizability of unions of spaces. Proc. Amer. Math. Soc., **10**, 361-366 (1959).
[9] H. Tamano: Note on paracompactness. J. Math. Kyoto Univ., **3**, 137-143 (1963).