# 154. On Some Generalised Solution of a Nonlinear First Order Hyperbolic Partial Differential Equation

By Takaaki NISHIDA

(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1967)

We consider the following Cauchy problem in $t\geq 0$, $-\infty<x<+\infty$.

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=h(u) \tag{1}$$

$$u(0,x)=u_0(x), \tag{2}$$

where $f(u)\in C^2$, $h(u)\in C^1$ and $u_0(x)\in L^\infty$.
First we assume that $f_{uu}(u)\geq\delta>0$ for $\forall u$.

Oleinik [1] proved the uniqueness and existence theorem of the generalised solution for Cauchy problem $u_t+f(t,x,u)_x=g(t,x,u)$ with (2) under the condition $f_{uu}\geq$ const. $>0$ and $|g_u(t,x,u)|\leq$ const. Here we consider the case that $|g_u(t,x,u)|\leq$ const. is not satisfied and see that the uniqueness and existence theorem is valid for some case under the following definition of the generalised solution.

We call $u(t,x)$ the generalised solution of (1)(2), which satisfies the following:

i) $u(t,x)$ is a measurable and locally bounded function.

ii) for arbitrary continuously differentiable function $\varphi(t,x)$ with compact support

$$\iint_{t\geq 0}\left[u\frac{\partial\varphi}{\partial t}+f(u)\frac{\partial\varphi}{\partial x}+h(u)\varphi\right]dt\,dx+\int_{-\infty}^{+\infty}\varphi(0,x)u_0(x)dx=0. \tag{3}$$

iii)

$$\frac{u(t,x_1)-u(t,x_2)}{x_1-x_2}<K(t,x_1,x_2), \tag{4}$$

where $K(t,x_1,x_2)$ is continuous in $t>0$, $-\infty<x_1, x_2<+\infty$.

§1. **Uniqueness Theorem.** We have the following uniqueness theorem.

**Theorem.** *The generalised solution $u(t,x)$ of (1)(2) is unique under the following estimate.*

$$\begin{aligned} -\beta(t)\leq u(t,x)\leq\alpha(t,x) \quad &\text{for}\quad t\geq 0,\quad x\geq 0,\\ -\alpha(t,-x)\leq u(t,x)\leq\beta(t) \quad &\text{for}\quad t\geq 0,\quad x\leq 0, \end{aligned} \tag{5}$$

*where $\beta(t)$, $\alpha(t,x)$ are nonnegative and continuous in $\{t\geq 0\}$, $\{t\geq 0, x\geq 0\}$ respectively.*

This can be proved by a slight modification of the argument in [1] th. 1. Following it, let us assume that there exist two generalised solutions $u_1(t,x)$ $u_2(t,x)$ satisfying (5). It is sufficient to see that for any $F(t,x)\in C^1$ such that there exist $T>a>0$, $X>0$ (may depend on $F$) and $F\equiv 0$ for $t\geq T$, $0\leq t\leq \alpha$ or $|x|\geq X$, the following is true

$$(6)\qquad \iint_{t>0} F(t,x)(u_1(t,x)-u_2(t,x))dt\,dx=0.$$

For the proof of (6) we consider the dual linear differential equation

$$(7)\qquad \frac{\partial v}{\partial t}+\phi(t,x)\frac{\partial v}{\partial x}+\psi(t,x)v=F(t,x),$$

where $\phi(t,x)=\dfrac{f(u_1)-f(u_2)}{u_1-u_2}$, $\psi(t,x)=\dfrac{h(u_1)-h(u_2)}{u_1-u_2}$. In order to have a continuously differentiable solution $v(t,x)$ we take the following averaged equation instead of (7).

$$(8)\qquad \frac{\partial v}{\partial t}+\phi_h(t,x)\frac{\partial v}{\partial x}+\psi_{h\rho}(t,x)v=F(t,x),$$

where $\phi_h$, $\psi_{h\rho}$ are averaged functions of $\phi$, $\psi$ and $\psi_{h\rho}=0$ for $0\leq t\leq \rho$, therefore $h\to 0$, $\rho\to 0$ include $\phi_h\to\phi$, $\psi_{h\rho}\to\psi$.

We take as the boundary condition for $v(t,x)$ the following:

$$(9)\qquad \begin{array}{ll} v(T,x)=0 & \text{for } -\infty<x<+\infty, \\ v(t,\pm\infty)=0 & \text{for } 0\leq t\leq T, \end{array}$$

where it is sufficient for the latter to take $v(t,\pm(X+CT))=0$ for each $F(t,x)$, where $C=\max_{0\leq t\leq T}\beta(t)$.

Now we have

$$\begin{array}{ll} -C\leq \phi_h(t,x)\leq A(x) & \text{for } x\geq 0,\quad 0\leq t\leq T, \\ -A(-x)\leq \phi_h(t,x)\leq C & \text{for } x<0,\quad 0\leq t\leq T, \\ |\psi_{h\rho}(t,x)|\leq \text{const.} & \text{for } (t,x)\in D, \end{array}$$

where $D=\{(t,x)\,0\leq t\leq T, |x|\leq X\}$ and $A(x)\in C^1$ in $[0,+\infty)$.

Taking account of the explicit formula

$$(10)\qquad v(t_1,x_1)=\int_{t_0}^{t_1}F(s,x(s,t_1,x_1))\left[\exp\int_{t_1}^{s}-\psi_{h\rho}(\tau,x(t_1,x_1))d\tau\right]ds,$$

where $x(t,t_1,x_1)$ is the characteristics passing through the point $(t_1,x_1)$ i.e., the solution of $dx/dt=\phi_h(t,x)$ and $|x(t_0,t_1,x_1)|\geq X+CT$ or $t_0=T$, and also the fact that $F(t,x)=0$ for $(t,x)\notin D_\cap\{t\geq\alpha\}$, quite analogously to [1], we have the following:

$$v(t_1,x_1)\in C^1(0\leq t_1\leq T,\ -\infty<x_1<+\infty),$$

$$|v(t_1,x_1)|\leq \text{const. (indep. of } h\text{, dep. on } D \text{ and } \sup_D\{|u_1|,|u_2|\}),$$

$$\left|\frac{\partial v}{\partial x_1}\right|\leq \text{const. (indep. of } h) \text{ for } (t_1,x_1)\in D_\cap\{t\geq\alpha\},\ \rho\text{: fixed},$$

$$\underset{-\infty<x_1<+\infty}{\text{Variation}}\ v(t_1,x_1)<\text{const. (indep. of } h \text{ and } t) \text{ for } \rho\text{: fixed}.$$

Thus using the definition (3) of the generalised solution, and tending $h$, $\rho$ to zero appropriately, then we have

$$\text{(11)}\quad \iint_{t\geq 0}(u_1-u_2)F\,dt\,\mathrm{d}x=\iint(u_1-u_2)\Big(\frac{\partial v}{\partial t}+\phi_h\frac{\partial v}{\partial x}+\psi_{h\rho}v\Big)dt\,dx$$
$$=\iint(u_1-u_2)\Big[(\phi_h-\phi)\frac{\partial v}{\partial x}+(\psi_{h\rho}-\psi)v\Big]dt\,dx=0.$$

§ 2. **Existence Theorem.** Hereafter we assume that

$$\text{(12)}\quad \begin{aligned} &h(u)\leq \text{const. } (u^2+1) \quad \text{for} \quad u\geq 0,\\ &h(u)\geq -\text{const. } (u^2+1) \quad \text{for} \quad u<0 \text{ and there exist}\\ &u_0=\text{ const. such that } h(u_0)=0.\end{aligned}$$

For the initial value $u_0(x)$ we assume

$$\text{(13)}\quad u_0(x)-u_0$$

is some bounded measurable function with compact support.

**Theorem.** *The generalised solution of Cauchy problem* (1) (2), *which satisfies the estimate* (5), *exists for* $0\leq t<+\infty$, $-\infty<x<+\infty$ *under the assumptions* (12) (13).

The proof is analogous to that of [2]. For the simplicity of the argument we discuss the case that $f_u(0)=0$, $h(0)=0$ and $u_0(x)$ is a $L^\infty$-function with compact support.

The solution of the characteristic equation of (1)

$$\text{(14)}\quad \frac{dx}{dt}=f_u(u), \qquad \frac{du}{dt}=h(u)$$

with the initial values $x(0, \xi)=\xi$, $u(0, \xi)=u_0(\xi)$, $\xi\in(a, b)$ satisfies the following

$$\text{(15)}\quad \begin{aligned}&\int_{u_0(\xi)}^{u(t,\xi)}\frac{f_u(u)}{h(u)}du=x(t, \xi)-\xi, \text{ or}\\ &u(t, \xi)=u_0(\xi) \quad \text{for} \quad h(u_0(\xi))=0.\end{aligned}$$

Because of the continuous differentiability of $h(u)$ and the formula (15) with $f_{uu}\geq\delta>0$ we have

$$\text{(16)}\quad \begin{aligned}&0\leq u(t, \xi)\leq u_0(\xi)e^{x-\xi} \quad \text{for} \quad u_0(\xi)\geq 0,\\ &0\geq u(t, \xi)\geq u_0(\xi)e^{-(x-\xi)} \quad \text{for} \quad u_0(\xi)<0.\end{aligned}$$

Differentiation (14) with respect to $\xi$ gives

$$\frac{\partial u(t, \xi)}{\partial \xi}=\frac{\partial u(0, \xi)}{\partial \xi}\exp\int_0^t h_u(u(\tau, \xi))d\tau$$
$$\frac{\partial x(t, \xi)}{\partial \xi}=\frac{\partial x(0, \xi)}{\partial \xi}+\frac{\partial u(0, \xi)}{\partial \xi}\int_0^t f_{uu}(u(\tau, \xi))\exp\int_0^\tau h_u(u(\eta, \xi))d\eta d\tau$$

and so under the condition $\partial x(0, \xi)/\partial\xi\geq 0$, $\partial u(0, \xi)/\partial\xi\geq 0$ we have

$$\text{(17.1)}\quad \frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}\Big/\frac{\partial x}{\partial \xi}\leq\frac{\exp\int_0^t h_u(u(\tau, \xi))d\tau}{\int_0^t f_{uu}(u(\tau, \xi))\exp\int_0^\tau h_u(u(\eta, \xi))d\eta d\tau}$$
$$\leq\frac{\text{const.}\exp|x|}{\delta t}.$$

In the case that the initial is $x(0, \varepsilon)=\text{const}$,

$$u(0,\varepsilon)=u_-+\varepsilon(u_+-u_-) \quad \text{for} \quad 0\leq\varepsilon\leq 1,\ u_-<u_+$$

the same is true:

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \varepsilon}\Big/\frac{\partial x}{\partial \varepsilon}\leq\frac{\text{const.}\exp|x|}{\delta t}. \tag{17.2}$$

Now we approximate the initial condition $u_0(x)$ with piecewise constant functions $u^h(0,x)(h>0)$:

$$u^h(0,x)=\frac{1}{h}\int_{kh}^{(k+1)h}u_0(\xi)d\xi \quad \text{for} \quad kh<x<(k+1)h, \tag{18}$$

where $u^h(0,x)\equiv 0$ for $|x|\geq{}^{\exists}N>0$,

then we construct the generalised solution for the Cauchy problem (1)(18) by means of the solution of the characteristics equation (14) with the initial values

$$x(0,\xi)=\xi, \qquad u(0,\xi)=u^h(0,\xi), \tag{19}$$

where

$$\xi\in(kh,(k+1)h),\ k=0,\pm 1,\pm 2,\cdots,$$

and if $u^h(0,kh\text{-}0)<u^h(0,kh+0)$, then we supplement the initial values

$$x(0,\varepsilon)=kh,\ u(0,\varepsilon)=u^h(0,kh-0)+\varepsilon(u^h(0,kh+0)-u^h(0,kh-0)), \tag{20}$$

for $0\leq\varepsilon\leq 1$ and necessary $k$.

The method to construct the generalised solutions $u^h(t,x)$ for the Cauchy problem (1)(18) is analogous to that of [2], thus using the formulae (16), (17) for $u^h(t,x)$ we have the following

$$-C\leq u^h(t,x)\leq\alpha(x) \quad \text{for } x\geq 0,$$

$$-\alpha(-x)\leq u^h(t,x)\leq C \quad \text{for} \quad x<0,$$

$$\frac{u^h(t,x_1)-u^h(t,x_2)}{x_1-x_2}<K(t,x_1,x_2),$$

where $K(t,x_1,x_2)$ is continuous in $t>0$, $-\infty<x_1,x_2<+\infty$. On these bases by the analogous argument in [2] we see that for ${}^{\forall}K$: compact subdomain in $\{t\geq 0, -\infty<x<+\infty\}$ $u^h(t,x)$ is uniformly bounded and compact in $L^1(K)$, i.e.,

$$\sup_K|u^h(t,x)|\leq\text{const. (indep. of } h),$$

$$\underset{-{}^{\forall}X\leq x\leq X}{\text{Variation}}\{u^h(t,x)\}\leq\frac{\text{const.}}{t}\ \text{(indep. of } h),$$

$$\int_{-X}^{X}|u^h(t_1,x)-u^h(t_2,x)|dx\leq\text{const.}\,|t_1-t_2|$$

$$\text{for} \quad t_1,t_2\geq{}^{\forall}\alpha>0$$

(indep. of $h$, dep. on $X$ and $\alpha$).

By the way $u^h(t,x)$ satisfies the following for any continuously differentiable function $\varphi(t,x)$ with compact support:

$$\iint_{t\geq 0}\left[u^h\frac{\partial\varphi}{\partial t}+f(u^h)\frac{\partial\varphi}{\partial x}+h(u^h)\varphi\right]dtdx+\int_{-\infty}^{+\infty}u^h(0,x)\varphi(0,x)dx=0, \tag{21}$$

and considering that there exist a subsequence $u^{h_j}(t,x)$ of $u^h(t,x)$ such that $u^{hj}(t, x) \to u(t, x) \in L^\infty(\mathrm{loc})$ in $L^1(\mathrm{loc})$, passing to the limit in (21) along $hj \to 0$, then we have the following for the limit function $u(t, x)$

$$\iint_{t\geq 0}\left[u\frac{\partial\varphi}{\partial t}+f(u)\frac{\partial\varphi}{\partial x}+h(u)\varphi\right]dt\,dx+\int_{-\infty}^{+\infty}\varphi(0, x)u_0(x)dx=0,$$

that is, $u(t, x)$ is the desired generalised solution for (1)(2), and also the uniqueness of the generalised solution concludes that all sequence $u^h(t, x)$ converge to $u(t, x)$ in $L^1(\mathrm{loc})$.

**Remark.** 1. For the equation $u_t+(u^2/2)_x=g(x)u^2$, where $g(x)$ is any continuously differentiable function, we have an analogous result to the above, if we take the above definition of the generalised solution. (cf. [2]).

2. If we take the assumption $u_0(x) \in L^\infty$ instead of

$$u_0(x)-u_0 \in L^\infty \cap \mathcal{S}'$$

(or $u_0(x) \in L^\infty$ and the set $\{x \mid h(u_0(x)) \neq 0\}$ is equivalent almost everywhere to some compact set), we can not generally expect that the generalised solution of (1)(2) exists in $t \geq 0$ and is locally bounded in $t \geq 0$, $-\infty < x < +\infty$ under the assumption (12).

3. If we take $h(u)=u^{2+\alpha}(\alpha>0$, const.), then even for the case $u_0(x) \in L^\infty \cap \mathcal{S}'$ we can not generally expect that the generalised solution of (1)(2) exists and is locally bounded in $t \geq 0$, $-\infty<x<+\infty$, but the same existence theorem as the above is true under the additional condition that if $u$ is infinite, then

$$\int_{u_0}^{u}\frac{f_u(v)}{h(v)}dv \text{ is infinite for finite values } u_0.$$

(by virtue of (15)).

The writer wishes to express his sincere gratitude to Professor M. Tada and Professor M. Yamaguti for valuable encouraging suggestions and their interest in this note.

## References

[1] O. A. Oleinik: Discontinuous solutions of nonlinear differential equations. Amer. Math. Soc. Trans. Ser. II, **26**, 95-172.

[2] T. Nishida: A remark on a nonlinear hyperbolic equation of first order. Mem. Fac. Eng. Kyoto Univ., **24**, 213-223 (1967).