193. On Free Abelian m-Groups. I

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In this article, the notions of free abelian m-group and the tensor product of abelian m-groups will be introduced and their more immediate properties are developed.

Recall that

Definition. An algebraic system $(M, [\ \ ])$ or simply M is called an *m-semigroup* if and only if $[\ \ ]: M^m \to M$ satisfies the m-associative law, i.e.

$[[x_1x_2\cdots x_m]x_{m+1}\cdots x_{2m-1}]=[x_1x_2\cdots x_i[x_{i+1}x_{i+2}\cdots x_{i+m}]x_{i+m+1}\cdots x_{2m-1}]$
for each $i=1, 2, \cdots, m-1$ and all $x_1, x_2, \cdots, x_{2m-1} \in M$.

The m-ary operation $[\ \ ]$ can be extended in a natural way to an n-ary operation, where n is greater than m and such that $n\equiv 1 \pmod{m-1}$. This is done by defining

$$[x_1x_2\cdots x_n]=[\cdots[[x_1x_2\cdots x_m]x_{m+1}\cdots x_{2m-1}]\cdots x_n]$$

for all $x_1, x_2, \cdots, x_n \in M$ and $n\equiv 1 \pmod{m-1}$. The following generalized associative law holds for m-semigroups (see R. H. Bruck [2]):

$$[x_1x_2\cdots x_m]=[x_1x_2\cdots x_i[x_{i+1}x_{i+2}\cdots x_j]x_{j+1}\cdots x_n]$$

for $n\equiv 1 \pmod{m-1}$, $1<j-i\equiv 1 \pmod{m-1}$, and all $x_1, x_2, \cdots, x_n \in M$.

For convenience, one may designate $\langle k\rangle=k(m-1)+1$ and $x^{\langle k\rangle}=[x_1x_2\cdots x_{\langle k\rangle}]$ with $x_1=x_2=\cdots=x_{\langle k\rangle}=x$. Observe that the following exponential laws hold in any m-semigroup: (1) $(x^{\langle h\rangle})^{\langle k\rangle}=x^{\langle hk(m-1)+h+k\rangle}$ and (2) $[x^{\langle k_1\rangle}x^{\langle k_2\rangle}\cdots x^{\langle k_m\rangle}]=x^{\langle k_1+k_2+\cdots+k_m+1\rangle}$.

Definition. An $(m-1)$-tuple $(u_1, u_2, \cdots, u_{m-1})$ of elements from an m-semigroup $(M, [\ \ ])$ is called an $(m-1)$-*adic identity* of M if and only if $[xu_1u_2\cdots u_{m-1}]=x=[u_1u_2\cdots u_{m-1}x]$ for all $x\in M$. In a similar manner, for any $n\equiv 1 \pmod{m-1}$, the notion of a $(n-1)$-adic identity of M may be defined.

Note that $(u_1, u_2, \cdots, u_{k(m-1)})$ is a $k(m-1)$-adic identity if and only if $([u_1u_2\cdots u_{(k-1)(m-1)}u_{(k-1)(m-1)+1}], \cdots, u_{k(m-1)})$ is an $(m-1)$-adic identity.

Definition. An m-semigroup $(M, [\ \ ])$ is an *m-group* if and only if

(a) for $u_1, u_2, \cdots u_{m-2} \in M$, there exists a $u\in M$ such that $(u_1, u_2, \cdots, u_{m-2}, u)$ is an $(m-1)$-adic identity of M;

(b) for $u_1, u_2, \cdots u_{m-2} \in M$, there exists a $u\in M$ such that $(u, u_1, \cdots, u_{m-2})$ is an $(m-1)$-adic identity of M.

Observe that if $[u_1u_2\cdots u_{m-1}a_1]=a_1$ for a fixed $a_1\in M$, so that for any $a_2,\cdots,a_{m-2}\in M$ there exists an $a_{m-1}\in M$ such that $(a_1,a_2,\cdots,a_{m-1})$ is an $(m-1)$-adic identity, then $[u_1u_2\cdots u_{m-1}x]=[u_1u_2\cdots u_{m-1}[a_1a_2\cdots a_{m-1}x]]=[[u_1u_2\cdots u_{m-1}a_1]a_2\cdots a_{m-1}x]=[a_1a_2\cdots a_{m-1}x]=x$ for all $x\in M$. From this it follows that if $(u_1,u_2,\cdots,u_{m-2},u)$ is an $(m-1)$-adic identity, so that, in particular, $[uu_1\cdots u_{m-2}u]=u$, then $[uu_1\cdots u_{m-2}x]=x$ for all $x\in M$. Hence, if $(u_1,u_2,\cdots,u_{m-2},v)$ is another $(m-1)$-identity, then $u=[uu_1\cdots u_{m-2}v]=v$. In exactly the same manner, this time using (b) in the definition, if $(u',u_1,\cdots,u_{m-2})$ and $(v',u_1,\cdots,u_{m-2})$ are both $(m-1)$-adic identities then $u'=v'$. Suppose, now, that $(u_1,\cdots,u_{m-2},u)$ and $(u',u_1,\cdots,u_{m-2})$ are both $(m-1)$-adic identities. As we have previously shown, the first of these implies that $[uu_1\cdots u_{m-2}x]=x$ for all $x\in M$, while the second implies $[xu_1\cdots u_{m-2}u']=x$ for all $x\in M$. Choosing $x=u'$ in the first and $x=u$ in the second, we obtain $u=[uu_1\cdots u_{m-2}u']=u'$. Finally, we have thus shown that for $u_1,\cdots,u_{m-2}\in M$, there exists uniquely a $u=(u_1,\cdots,u_{m-2})^{-1}\in M$ such that both $(u_1,\cdots,u_{m-2},u)$ and $(u,u_1,\cdots,u_{m-2})$ are $(m-1)$-adic identities. This also proves that our definition of an m-group is equivalent to that of D. Boccioni [1].

Incidentally, the above results also show that an m-group may be defined as an algebraic system $(M,[\quad],(\quad)^{-1})$ such that $(M,[\quad])$ is an m-semigroup and $(\quad)^{-1}$: $M^{m-2}\to M$ is an $(m-1)$-ary operation such that $[x_1x_2\cdots x_{m-2}(x_1,x_2,\cdots,x_{m-2})^{-1}x]=[(x_1,x_2,\cdots,x_{m-2})^{-1}x_1x_2\cdots x_{m-2}x]=[xx_1\cdots x_{m-2}(x_1,x_2,\cdots,x_{m-2})^{-1}]=[x(x_1,x_2,\cdots,x_{m-2})^{-1}x_1\cdots x_{m-2}]=x$ for all $x,x_1,x_2,\cdots,x_{m-2}\in M$. Whence

Theorem 1. *The family of all m-groups is an equational or primitive class.*

One further concludes from the above discussions that if $(u_1,\cdots,u_{m-2},u_{m-1})$ is an $(m-1)$-adic identity, then $(u_{m-1},u_1,\cdots,u_{m-2})$ is also an $(m-1)$-adic identity. By iteration, we obtain the result that $(u_{\sigma(1)},\cdots,u_{\sigma(m-2)},u_{\sigma(m-1)})$ is an $(m-1)$-adic identity for all powers σ of the permutation $(12\cdots m-1)$.

As an example of an m-group, consider the following. Let $X_1,X_2,\cdots,X_{m-1}$ be sets of the same cardinality. Denote by $S(X_1,X_2,\cdots,X_{m-1})$ the collection of all one-to-one functions f on $\bigcup_{i=1}^{m-1}X_i$ onto itself such that $f(X_i)=X_{\sigma(i)}$ for all $i=1,\cdots,m-1$, where σ is the cyclic permutation $(12\cdots m-1)$. Under the operation defined by

$$[f_1f_2\cdots f_m]=f_1\circ f_2\circ\cdots\circ f_m,$$

$S(X_1,X_2,\cdots,X_{m-1})$ is clearly an m-semigroup. If $f_1,f_2,\cdots,f_{m-2}\in S(X_1,X_2,\cdots,X_{m-1})$, then $f_1\circ f_2\circ\cdots\circ f_{m-2}=f$ is a one-to-one function such that $f(S_1)=S_{m-1},f(S_2)=S_1,\cdots,f(S_{m-1})=S_{m-2}$ and hence

$f^{-1} \in S(X_1, X_2, \cdots, X_{m-2})$. Both $(f^{-1}, f_1, \cdots, f_{m-2})$ and $(f_1, \cdots, f_{m-2}, f^{-1})$ are m-adic identities of $S(X_1, X_2, \cdots, X_{m-1})$.

For self-containment, we shall state and prove the following two results which will be used later.

Theorem 2. *Every m-group $(M, [\ \ ])$ is isomorphic to an m-group of functions.*

Proof. For each $i=1, \cdots, m-1$, defined a relation $\overset{i}{\sim}$ on the cartesian product $M\times M\times \cdots \times M$ (i-times) such that $(a_1, a_2, \cdots, a_i) \overset{i}{\sim} (b_1, b_2, \cdots, b_i)$ if and only if $[a_1a_2 \cdots a_ix_{i+1} \cdots x_m]=[b_1b_2 \cdots b_ix_{i+1} \cdots x_m]$ for all $x_{i+1}, x_{i+2}, \cdots, x_m \in M$. Observe that if $[a_1a_2 \cdots a_ic_{i+1} \cdots c_m]=[b_1b_2 \cdots b_ic_{i+1} \cdots c_m]$ for some fixed $c_{i+1}, \cdots, c_m \in M$, so that $(c_2, c_3, \cdots, c_m)$ is an $(m-1)$-adic identity, then

$$\begin{aligned}[a_1a_2 \cdots a_ix_{i+1} \cdots x_m] &= [a_1a_2 \cdots a_i[c_{i+1} \cdots c_mc_2 \cdots c_ix_{i+1}] \cdots x_m] \\ &= [[a_1a_2 \cdots a_ic_{i+1} \cdots c_m]c_2 \cdots c_ix_{i+1} \cdots x_m] \\ &= [[b_1b_2 \cdots b_ic_{i+1} \cdots c_m]c_2 \cdots c_ix_{i+1} \cdots x_m] \\ &= [b_1b_2 \cdots b_i[c_{i+1} \cdots c_mc_2 \cdots c_ix_{i+1}]x_{i+2} \cdots x_m] \\ &= [b_1b_2 \cdots b_ix_{i+1} \cdots x_m]\end{aligned}$$

for all $x_{i+1}, \cdots, x_m \in M$.

It is easy to see that $\overset{i}{\sim}$ for each $i=1, 2, \cdots, m-1$ is an equivalence relation on $M\times M\times \cdots \times M$ (i times). Set $X_i=M\times M\times \cdots \times M/\overset{i}{\sim}$. Note that $M/\overset{1}{\sim}=X_1=M$ since $(a)\overset{1}{\sim}(b)$ if and only if $a=b$. Now, consider the transformation m-group $S(X_1, X_2, \cdots, X_{m-1})$. For each $x \in M$, define f_x by $f_x((x_1, \cdots, x_i)/\overset{i}{\sim})=(x_1, \cdots, x_i, x)/\overset{i+1}{\sim}$ for $i=1, 2, \cdots, m-2$ and $f_x((x_1, \cdots, x_{m-1})/\overset{m-1}{\sim})=[x_1x_2 \cdots x_{m-1}x]/\overset{1}{\sim}$ for $i=m-1$. Suppose $f_x((x_1, \cdots x_i)/\overset{i}{\sim})=f_x((y_1, \cdots, y_i)/\overset{i}{\sim})$ so that $(x_1, \cdots, x_i, x)\overset{i+1}{\sim}(y_1, \cdots, y_i, x)$. This means that for all $x_{i+2}, \cdots, x_m \in M$ we have $[x_1 \cdots x_ixx_{i+2} \cdots x_m]=[y_1 \cdots y_ixx_{i+2} \cdots x_m]$ and hence $(x_1, \cdots, x_i)\overset{i}{\sim}(y_1, \cdots, y_1)$ or $(x_1, \cdots, x_i)/\overset{i}{\sim}=(y_1, \cdots, y_i)/\overset{i}{\sim}$. Thus, f_x is one-to-one. To show that f_x is onto, consider any $(a_1, \cdots, a_{i+1})/\overset{i+1}{\sim}$. Choose $a_{i+2}, \cdots, a_{m-1} \in M$ such that $(a_1, \cdots, a_{m-1})$ is an $(m-1)$-adic identity and choose $b_1, \cdots, b_i \in M$ such that $(x, a_{i+2}, \cdots, a_{m-1}, b_1, \cdots, b_i)$ and hence $(b_1, \cdots, b_i, x, a_{i+2}, \cdots, a_{m-1})$ is an $(m-1)$-adic identity. Then $f_x((b_1, \cdots, b_i)/\overset{i}{\sim})=(b_1, \cdots, b_i, x)/\overset{i+1}{\sim}=(a_1, \cdots, a_{i+1})/\overset{i+1}{\sim}$. A variation of this argument will show that in general f_x is onto.

Define $h: M\rightarrow S(X_1, X_2, \cdots, X_{m-1})$ by $h(x)=f_x$. If $f_x=f_y$ and $(u_1, \cdots, u_{m-1})$ is any $(m-1)$-adic identity, then $x=[u_1 \cdots u_{m-1}x]=[u_1\cdots u_{m-1}y]=y$, that is, h is one-to-one. Moreover, $h([x_1x_2\cdots x_m])=f_{[x_1x_2\cdots x_m]}=f_{x_1}\circ f_{x_2}\circ \cdots \circ f_{x_m}=[f_{x_1}f_{x_2} \cdots f_{x_m}]=[h(x_1)h(x_2) \cdots h(x_m)]$ for all $x_1, x_2, \cdots, x_m \in M$.

Theorem 3 (*Post Coset Theorem*). *Every m-group M is a coset $Nx=xN=M$ of a (2-group) group G by a normal subgroup $N=M^{m-1}$ whose index is a divisor of $m-1$. Moreover, G/N is a cyclic group generated by M and $G=M\cup M^2\cup\cdots\cup M^{m-1}$.*

Proof. By Theorem 2, M is isomorphic to and hence may be identified with a subset of the symmetric group $S(\bigcup_{i=1}^{m-1}X_i)$ of all one-to-one transformations of the set $\bigcup_{i=1}^{m-1}X_i$ onto itself. The operation in M is an extension of the operation of composition in this group. Let G be the least subgroup of $S(\bigcup_{i=1}^{m-1}X_i)$ containing M. Since $M^m=M$, then note that $G=M\cup M^2\cup\cdots\cup M^{m-1}$. If $g\in G$ such that $g=x_1x_2\cdots x_i$ for $x_1,\cdots,x_i\in M$ and $i\leqq m-2$, then $g^{-1}=x_{i+1}\cdots x_{m-1}$ for $x_{i+1},\cdots,x_{m-1}\in M$ with the property that $x_1x_2\cdots x_{m-1}=1$ or $(x_1,x_2,\cdots,x_{m-1})$ is an $(m-1)$-adic identity. Hence $gM^{m-1}g^{-1}\subseteqq M^iM^{m-1}M^{m-i-1}=M^{m-1}$. On the other hand, if $g\in G$ and $g\in M^{m-1}$ so that $g^{-1}\in M^{m-1}$, then $gM^{m-1}g^{-1}\subseteqq M^{m-1}M^{m-1}M^{m-1}=M^{m-1}$. Thus, M^{m-1} is a normal subgroup of G. Now, if $x_1x_2\cdots x_{m-2}x=1$ or $(x_1,x_2,\cdots,x_{m-2},x)$ is an $(m-1)$-adic identity, then $M=Mx_1x_2\cdots x_{m-2}x\subseteqq M^{m-1}x\subseteqq M^m=M$. Whence $M^{m-1}x=M$. The rest of the conclusions follow.