191. *Some Properties of M-Spaces*

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In a previous paper [9] we have introduced the notion of M-spaces. A topological space X is an M-space if there exists a normal sequence $\{\mathfrak{U}_i \mid i=1, 2, \cdots\}$ of open coverings of X satisfying condition (M) below:

$$(M)\quad \begin{cases} \text{If } \{K_i\} \text{ is a sequence of non-empty subsets of } X \text{ such that} \\ K_{i+1}\subset K_i,\ K_i\subset \mathrm{St}(x_0, \mathfrak{U}_i) \text{ for each } i \text{ and for some fixed point} \\ x_0 \text{ of } X, \text{ then } \cap \bar{K}_i\neq\phi. \end{cases}$$

Condition (M) is equivalent to the original condition (40) in [9]. In this paper we shall discuss some properties of M-spaces.

1. In [7] (cf. also J. Dugundji [2, p. 196]) we proved that a T_1-space X is metrizable if and only if there is a sequence $\{\mathfrak{F}_i\}$ of locally finite closed coverings of X such that for any point x and for any neighborhood V of x there is some i for which $\mathrm{St}(x, \mathfrak{F}_i)\subset V$. Thus it is natural to consider a topological space X such that there is a sequence $\{\mathfrak{F}_i\}$ of locally finite closed coverings of X satisfying condition (M). Such a space we shall call an M^*-space after T. Ishii [3]. Corresponding to our metrization theorem mentioned above, the following theorem holds.

Theorem 1.1. *If X is an M-space, then X is an M^*-space with property (C) below:*

$$(C)\quad \begin{cases} \text{For any locally finite collection } \{F_\lambda\} \text{ of closed sets of } X \text{ there} \\ \text{exists a locally finite collection } \{G_\lambda\} \text{ of open sets of } X \text{ such} \\ \text{that } F_\lambda\subset G_\lambda \text{ for each } \lambda. \end{cases}$$

In case X is normal, the converse is true.

Proof. Let X be an M-space; by [8, Theorem 1. 2] there is a normal sequence $\{\mathfrak{V}_i\}$ of locally finite open coverings of X satisfying condition (M), and hence $\{\mathfrak{F}_i\}$ satisfies condition (M) where we set $\mathfrak{F}_i=\{\bar{V} \mid V\in\mathfrak{V}_i\}$. Since by A. Okuyama [10] any M-space has property (C), the first part of the theorem is proved.

To prove the second part, let $\{\mathfrak{F}_i\}$ be a sequence of locally finite closed coverings of X satisfying condition (M); without loss of generality we may assume that $\mathfrak{F}_{i+1}$ is a refinement of $\mathfrak{F}_i$ for $i=1, 2, \cdots$. Let us set

$$(1)\qquad \mathfrak{L}_i=\{\mathrm{St}(F, \mathfrak{F}_i) \mid F\in\mathfrak{F}_i\},\qquad i=1, 2, \cdots.$$

Then, if $A\subset X$, we have

(2) $$\mathrm{St}(A, \mathfrak{L}_i)=\mathrm{St}(\mathrm{St}(\mathrm{St}(A, \mathfrak{F}_i), \mathfrak{F}_i), \mathfrak{F}_i).$$

Let K_i, $i=1, 2, \cdots$ be non-empty subsets of X such that

(3) $$K_{i+1}\subset K_i, \qquad K_i\subset \mathrm{St}(x_0, \mathfrak{L}_i), \qquad i=1, 2, \cdots,$$

where $x_0 \in X$. Then from (2) and (3) we get

(4) $$\mathrm{St}(\mathrm{St}(K_i, \mathfrak{F}_i), \mathfrak{F}_i)\cap \mathrm{St}(x_0, \mathfrak{F}_i)\neq\phi, \qquad i=1, 2, \cdots.$$

Since $\{\mathfrak{F}_i\}$ satisfies condition (M), there is a point $x_1 \in \cap\,\mathrm{St}(\mathrm{St}(K_i, \mathfrak{F}_i), \mathfrak{F}_i)$. Then

(5) $$\mathrm{St}(K_i, \mathfrak{F}_i)\cap \mathrm{St}(x_1, \mathfrak{F}_i)\neq\phi, \qquad i=1, 2, \cdots.$$

Applying condition (M), we see that there is a point $x_2 \in \cap\,\mathrm{St}(K_i, \mathfrak{F}_i)$. Then we have

(6) $$K_i\cap \mathrm{St}(x_2, \mathfrak{F}_i)\neq\phi, \qquad i=1, 2, \cdots.$$

By appealing to condition (M) again, we have $\cap\,\bar{K}_i\neq\phi$. Therefore $\{\mathfrak{L}_i\}$ satisfies condition (M).

Let $\mathfrak{F}_i=\{F_{i\lambda} \mid \lambda \in \wedge_i\}$. Then

$$\mathfrak{L}_i=\{\mathrm{St}(F_{i\lambda}, \mathfrak{F}_i) \mid \lambda \in \wedge_i\} \quad \text{and} \quad F_{i\lambda}\subset \mathrm{Int}\,(\mathrm{St}(F_{i\lambda}, \mathfrak{F}_i)).$$

Hence, if X has property (C), there is, for each i, a locally finite open covering $\mathfrak{G}_i=\{G_{i\lambda} \mid \lambda \in \wedge_i\}$ such that $F_{i\lambda}\subset G_{i\lambda}\subset \mathrm{Int}\,(\mathrm{St}(F_{i\lambda}, \mathfrak{F}_i))$, $\lambda \in \wedge_i$. If X is normal, there is a normal sequence $\{\mathfrak{U}_i\}$ of locally finite open coverings of X such that $\mathfrak{U}_i$ is a refinement of $\mathfrak{G}_i$ for each i. This completes our proof of the second part.

Theorem 1.2. *A collectionwise normal space is an M-space if and only if it is an M^*-space.*

Proof. Every normal M^*-space is countably paracompact as is shown by Ishii [3]. Hence the "if" part follows immediately from Theorem 1.1 by a theorem of M. Katětov [5].

By Theorem 1.2 every paracompact normal M^*-space is an M-space. Hence the problem raised by Ishii [3] is settled hereby.

2. A map $f: X\rightarrow Y$ is called a quasi-perfect map if it is a continuous closed surjective map such that $f^{-1}(y)$ is countably compact.

Lemma 2.1. *Let $f: X\rightarrow Y$ be a quasi-perfect map. If X is an M^*-space, so is Y. If X has property (C), so has Y.*

The first part of this lemma is proved by Ishii [3]. The proof of the second part is straightforward, since if $\{A_\lambda\}$ is locally finite collection of subsets of X then the collection $\{f(A_\lambda)\}$ of subsets of Y is also locally finite (cf. Okuyama [10]).

Now we are in a position to prove the following theorem, the first part of which is obtained by T. Ishii [3] and gives an affirmative answer to the problem posed by A. Arhangel'skii in a letter to the author: "Is the image of a paracompact p-space in the sense of [1] under a perfect map a p-space?" (For paracompact Hausdorff spaces p-spaces are identical with M-spaces).

Theorom 2.2. *Let $f: X\rightarrow Y$ be a quasi-perfect map. If X is* *an M-space and if either X or Y is normal, then Y is an M-space. If Y is an M-space, so is X.*

Proof. If X is normal, so is Y. Hence the first part follows readily from Theorem 1.1 and Lemma 2.1. Since the composite of two quasi-perfect maps is a quasi-perfect map, the second part is proved by [9, Theorem 6.1].

3. As an immediate consequence of Theorem 2.2 we obtain the following theorem, which was proved first by T. Kandô [4] under the assumption that each A_λ is G_δ, and later by J. Suzuki [11] generally.

Theorem 3.1. *Let $\{A_\lambda \mid \lambda \in \wedge\}$ be a locally finite closed covering of a topological space X. If A_λ is a normal M-space for each λ, so is X.*

Proof. Let P be the topological sum of disjoint spaces P_λ, $\lambda \in \wedge$ such that for each λ there is a homeomorphism h_λ from P_λ onto A_λ. If we define a map $h: P \to X$ by $h \mid P_\lambda = h_\lambda$ for $\lambda \in \wedge$ then h is a perfect map and P is a normal M-space. Hence by Theorem 2.2 X is a normal M-space.

4. In this section we shall show that there is a perfect map $f: X \to Y$ such that X is an M-space but Y is not, and such that X, Y are locally compact Hausdorff spaces. Thus Theorems 1.1 and 2.2 are not true unless the normality of X or Y is assumed.

Let ω_1 be the first uncountable ordinal and let us set

$$S = W(\omega_1+1) \times W(\omega_1+1) - (\omega_1, \omega_1)$$

where $W(\omega_1+1)$ is the set of all ordinals $\alpha \leqq \omega_1$ with the usual interval topology. In S, the sets

$$P = \{(\alpha, \omega_1) \mid \alpha < \omega_1\}, \qquad Q = \{(\omega_1, \beta) \mid \beta < \omega_1\}$$

are closed. Let X be the topological sum of disjoint spaces S_n, $n = 1, 2, \cdots$ such that for each n there is a homeomorphism φ_n of S onto S_n. Now we identify a point $\varphi_{2m-1}(p)$ with $\varphi_{2m}(p)$ for $p \in P$ and a point $\varphi_{2m}(q)$ with $\varphi_{2m+1}(q)$ for $q \in Q$. By this identification we have an identification space Y and an identification map $f: X \to Y$. Then f is a perfect map. Since X is locally compact Hausdorff, so is Y. Since S is countably compact, X is an M-space. But Y is not an M-space. To prove this, assume ψ to be a quasi-perfect map from Y onto some metric space T. Let us set $\theta_n = \psi \circ f \circ \varphi_n: S \to T$. Then $\theta_n(S)$ is compact and hence can be imbedded in the Hilbert cube $I^{\aleph_0}$. Therefore the map θ_n takes on the same constant value t_n on tails of P and Q. Since $\theta_{2m-1}(p) = \theta_{2m}(p)$ for $p \in P$ and $\theta_{2m}(q) = \theta_{2m+1}(q)$ for $q \in Q (m \geqq 1)$, we have $t_n = t_1$ for $n = 2, 3, \cdots$. Hence $\psi^{-1}(t_1) \cap f(S_n) \neq \phi$, $n = 1, 2, \cdots$. Therefore $\psi^{-1}(t_1)$ is not countably compact. Thus Y is not an M-space (but Y is a p-space in the sense of [1]).

If we set $C_i = \cup\{f(S_{2m+i}) \mid m=0, 1, \cdots\}$, $i=1, 2$, then $Y=C_1 \cup C_2$ and C_i are closed M-spaces. Thus Theorem 3.1 is not true if "normal" is removed from its statement.

5. The following generalizes [9, Theorem 6.2] and [4, Corollary 2 to Theorem 5].

Theorem 5.1. *Let* $A = \bigcap_{i=1}^{\infty} G_i$ *be a* G_δ*-set of a regular* M*-space* X*, where* G_i *are open sets. Then* A *is an* M*-space for cases* (a), (b), *and* (c) *below*: (a) A *is paracompact*; (b) *each* G_i *is paracompact*; (c) X *is normal and each* G_i *is an* F_σ*-set of* X.

Proof. Let $\{\mathfrak{U}_i\}$ be a normal sequence of locally finite open coverings of X satisfying condition (M). In case (a) there is a normal sequence $\{\mathfrak{V}_i\}$ of locally finite open coverings of A such that if $V \in \mathfrak{V}_i$ then $\bar{V} \subset G_i \cap U$ for some set U of $\mathfrak{U}_i$ ($i=1, 2, \cdots$). Then $\{\mathfrak{V}_i\}$ satisfies condition (M). In case (c) there are closed sets F_{ij}, $i, j=1, 2, \cdots$ such that $G_i = \cup\{F_{ij} \mid j=1, 2, \cdots\}$, and hence, if we select open sets L_{ij} such that $F_{ij} \subset L_{ij}$, $\bar{L}_{ij} \subset G_i$, then by [6, Theorem 3] the open covering $\{L_{ij} \mid j=1, 2, \cdots\}$ of G_i has a locally finite open refinement. Hence in cases (b) and (c) there is a normal sequence $\{\mathfrak{H}_{ij} \mid j=1, 2, \cdots\}$ of locally finite open coverings of G_i such that if $H \in \mathfrak{H}_{ij}$ then $\bar{H} \subset G_i$. Now let us set

$$\mathfrak{V}_i = \{(\bigcap_{k=1}^{i} H_k) \cap U \cap A \mid H_k \in \mathfrak{H}_{ki},\ k \leq i;\ U \in \mathfrak{U}_i\},\ i=1, 2, \cdots.$$

Then $\{\mathfrak{V}_i\}$ is a normal sequence of locally finite open coverings of A satisfying condition (M).

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