188. *Representation Ring of Lie Group F_4*

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1. **Introduction.** The aim of this paper is to determine the representation ring $R(F_4)$ of group F_4, which is a simply connected compact simple Lie group of exceptional type F. Let $\mathfrak{J}$ denote the Jordan algebra consisting of all 3-hermitian matrices over the division ring of Cayley numbers. The group F_4 is obtained as the automorphism group of $\mathfrak{J}$. Let $\mathfrak{J}_0$ be the set of all elements of $\mathfrak{J}$ with zero trace. Then $\mathfrak{J}_0$ is invariant by the operation of F_4. Thus we have an F_4-C-module $\mathfrak{J}_0\otimes_R C$.[1)] On the other hand, we know another F_4-C-module $F_4\otimes_R C$, where F_4 is the Lie algebra of F_4. The result is as follows: *$R(F_4)$ is a polynomial ring $Z[\lambda_1, \lambda_2, \lambda_3, \mu]$ with 4 variables $\lambda_1, \lambda_2, \lambda_3$, and μ, where λ_i is the class of the exterior F_4-C-module $\Lambda^i(\mathfrak{J}_0\otimes_R C)$ in $R(F_4)$ for $i=1, 2, 3$, and μ is the class of $\mathfrak{F}_4\otimes_R C$ in $R(F_4)$.* In this paper, we shall describe the outline of our methods; these may be analogous to those as in the cases of classical groups [1] and of group G_2 [2]. The details will appear in the Journal of the Faculty of Science, Shinshu University, vol. 3, 1968.

2. **Representation ring.** Let G be a topological group. Let $M(G)$ denote the set of all G-C-isomorphism classes of G-C-modules. The direct sum $V\oplus W$ and the tensor product $V\otimes W$ of two G-C-modules V, W define a semiring structure on $M(G)$. The representation ring $R(G)=(R(G), \phi)$ (where ϕ: $M(G)\to R(G)$ is a semiring homomorphism) is the universal ring associated with the semiring $M(G)$.

3. **Jordan algebra $\mathfrak{J}$, group F_4 and Lie algebra $\mathfrak{F}_4\otimes_R C$.**

Let $\mathfrak{C}$ denote the division ring of Cayley numbers and $\mathfrak{J}$ be the set of all 3-hermitian matrices X over $\mathfrak{C}$. In $\mathfrak{J}$, we define a Jordan multiplication by

$$X\circ Y=\frac{1}{2}(XY+YX).$$

Then $\mathfrak{J}$ is a 27-dimensional commutative distributive (non-associative) algebra over R. Let F_4 denote the group of all automorphisms of $\mathfrak{J}$. As is well known, F_4 is a simply connected compact simple Lie group of exceptional type F. Obviously, $\mathfrak{J}$ is an F_4-R-module.

1) R and C are the fields of real and complex numbers, respectively.

Let $\mathfrak{J}_0$ be the set of all elements of $\mathfrak{J}$ with zero trace. $\mathfrak{J}_0$ is a 26-dimensional R-submodule of $\mathfrak{J}$. Since each $x \in F_4$ invaries the trace of every $X \in \mathfrak{J}$, $\mathfrak{J}_0$ is also an F_4-R-module and $\mathfrak{J}$ is decomposable into the direct sum of R (with trivial group action) and $\mathfrak{J}_0$: $\mathfrak{J} = R \oplus \mathfrak{J}_0$. Thus we have an F_4-C-module $\mathfrak{J}_0 \otimes_R C$.

Let $\mathfrak{F}_4$ denote the Lie algebra of F_4, which consists of all R-homomorphism $A: \mathfrak{J} \to \mathfrak{J}$ satisfying

$$A(X \circ Y) = A(X) \circ Y + X \circ A(Y) \qquad \text{for } X, Y \in \mathfrak{J}.$$

$\mathfrak{F}_4$ is a 52-dimensional F_4-R-module by the group operation

$$(xA)(X) = x(A(x^{-1}(X))) \qquad \text{for } x \in F_4,\ A \in \mathfrak{F}_4,\ X \in \mathfrak{J}.$$

Thus we have an F_4-C-module $\mathfrak{F}_4 \otimes_R C$.

4. Maximal torus T and Weyl group W of F_4.

F_4 has three subgroups of type Spin (9): $\mathrm{Spin}^{(1)}(9)$, $\mathrm{Spin}^{(2)}(9)$, $\mathrm{Spin}^{(3)}(9)$, and has a subgroup Spin (8). That is,

$$\mathrm{Spin}^{(i)}(9) = \{x \in F_4 \mid x(E_i) = E_i\} \qquad \text{for } i = 1, 2, 3, \text{ where}$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad \text{and} \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

And $\mathrm{Spin}(8) = \mathrm{Spin}^{(1)}(9) \cap \mathrm{Spin}^{(2)}(9) \cap \mathrm{Spin}^{(3)}(9)$. Since the ranks of F_4 and Spin (8) are both 4, we choose a maximal torus T of F_4 in Spin (8).

The Weyl group $W = W(F_4)$ of F_4 is $N_T(F_4)/T$, where $N_T(F_4)$ is the normalizer of T in F_4. Each element $x \in N_T(F_4)$ induces a permutation of E_1, E_2, E_3. It follows that $W(F_4)$ is a semidirect product of W(Spin (8)) (the Weyl group of Spin (8)) and $\mathfrak{S}_3$ (the symmetric group of 3 factors).

5. Decompositions of $\mathfrak{J}_0 \otimes_R C$ and $\mathfrak{F}_4 \otimes_R C$.

Let $j: T \to F_4$ denote the inclusion. Then j induces the inclusion $R(F_4) \subset R(T)^W$, where $R(T)^W$ is the subring of $R(T)$ which is invariant by the Weyl group W.

The representation ring $R(T)$ of T is

$$R(T) = [\alpha_1, \alpha_1^{-1}, \alpha_2, \alpha_2^{-1}, \alpha_3, \alpha_3^{-1}, \alpha_4, \alpha_4^{-1}, \alpha_1^{1/2}\alpha_2^{1/2}\alpha_3^{1/2}\alpha_4^{1/2}]$$

and we have $R(\mathrm{Spin}(8)) = Z[\sigma, \tau, \Delta^-, \Delta^+]$ (cf. [1]), where

$$\sigma = \sum_{i=1}^{4} (\alpha_i + \alpha_i^{-1}),$$

$$\tau = \sum_{1 \leq i < j \leq 4} (\alpha_i + \alpha_i^{-1})(\alpha_j + \alpha_j^{-1}),$$

$$\Delta^- = \sum_{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = -1} \alpha_1^{\varepsilon_1/2}\alpha_2^{\varepsilon_2/2}\alpha_3^{\varepsilon_3/2}\alpha_4^{\varepsilon_4/2}$$

$$\Delta^+ = \sum_{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = 1} \alpha_1^{\varepsilon_1/2}\alpha_2^{\varepsilon_2/2}\alpha_3^{\varepsilon_3/2}\alpha_4^{\varepsilon_4/2}$$

($\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ are -1 or 1).

The main tools in the determination of $R(F_4)$ are the following decompositions of $\mathfrak{J} \otimes_R C$ and $\mathfrak{F}_4 \otimes_R C$ in $R(T)$.

(5.1) $$\lambda_1=\phi(\mathfrak{J}_0\otimes_R C)=2+\sigma+\Delta^-+\Delta^+,$$

(5.2) $$\mu=\phi(\mathfrak{J}_4\otimes_R C)=4+\sigma+\Delta^-+\Delta^++\tau.$$

Further by (5.1) we have

(5.3) $$\lambda_2=\phi(\Lambda^2(\mathfrak{J}_0\otimes_R C))=13+2(\sigma+\Delta^-+\Delta^+)+(\sigma\Delta^-+\Delta^-\Delta^++\Delta^+\sigma)+3\tau,$$

(5.4) $$\lambda_3=\phi(\Lambda^3(\mathfrak{J}_0\otimes_R C))=24+8(\sigma+\Delta^-+\Delta^+)+2\tau(\sigma+\Delta^-+\Delta^+)+3(\sigma\Delta^-+\Delta^-\Delta^++\Delta^+\sigma)+\sigma\Delta^-\Delta^++6\tau.$$

6. Ring structure of $R(T)^W$.

From (5.1)—(5.4) we have

(6.1) $$\begin{cases}\sigma+\Delta^-+\Delta^+=\lambda_1-2\\ \sigma\Delta^-+\Delta^-\Delta^++\Delta^+\sigma=\lambda_2+\lambda_1-3\mu-3\\ \sigma\Delta^-\Delta^+=\lambda_3-3\lambda_2-5\lambda_1+7\mu+2\lambda_1^2-2\lambda_1\mu+5\\ \tau=\mu-\lambda_1-2.\end{cases}$$

Note that the left side formulae in (6.1) are polynomials in $\lambda_1, \lambda_2, \lambda_3$, and μ.

Now, let f be a W-invariant polynomial. We know that any $W(\mathrm{Spin}(8))$-invariant polynomial is representable as a polynomial in $\sigma, \tau, \Delta^-, \Delta^+$, and Weyl group W is the semidirect product of $W(\mathrm{Spin}(8))$ and $\mathfrak{S}_3$ (which is the permutation group of 3 factors $\sigma, \Delta^-, \Delta^+$). Hence, f is a polynomial in $\sigma+\Delta^-+\Delta^+$, $\sigma\Delta^-+\Delta^-\Delta^++\Delta^+\sigma$, $\sigma\Delta^-\Delta^+$, and τ. Thus, by (6.1) f is representable as a polynomial in $\lambda_1, \lambda_2, \lambda_3$, and μ.

Next, we shall show that $\lambda_1, \lambda_2, \lambda_3$, and μ are algebraically independent. In fact, $\sigma, \Delta^-, \Delta^+$, and τ are algebraically independent, hence so are also $\sigma+\Delta^-+\Delta^+$, $\sigma\Delta^-+\Delta^-\Delta^++\Delta^+\sigma$, $\sigma\Delta^-\Delta^+$, and τ, and, therefore, by (5.1)—(5.4) $\lambda_1, \lambda_2, \lambda_3$, and μ are algebraically independent. Thus we have proved the following

Theorem. *The representation ring $R(F_4)$ of F_4 is a polynomial ring $[\lambda_1, \lambda_2, \lambda_3, \mu]$ with 4 variables $\lambda_1, \lambda_2, \lambda_3$, and μ.*

References

[1] J. Milnor: The representation rings of some classical groups. Notes for Mathematics, 402 (1963).

[2] I. Yokota: Representation ring of group G_2. Jour. Fac. Sci., Shinshu Univ., **2** (1967).