183. *An Important Relation in Homotopy Groups of Spheres*

By Hirosi TODA

Department of Mathematics, Kyoto University, Kyoto

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1. In computing the p-primary component ${}_pG_k$ of the k-stem group $G_k=\lim \pi_{n+k}(S^n)$, p denoting always an odd prime, an essential difficulty lies in the case $k=2p^2(p-1)-3$. Recently, Cohen [1] has announced that ${}_pG_{2p^2(p-1)-3}\approx Z_p$, which is equivalent to say that $\alpha_1\beta_1^p\neq 0$ for the generators α_1 of ${}_pG_{2p-3}$ and β_1 of ${}_pG_{2p(p-1)-2}$. The result of the present work, however, does not agree with this announcement. Our fundamental result is

Theorem. *For sufficiently large integer n, there exists a cell complex*

$$K=S^n\cup e^{n+2(p^2-1)(p-1)-1}\cup e^{n+2p^2(p-1)-1}\cup e^{n+2p^2(p-1)}$$

such that $\mathfrak{P}^{p^2}H^n(K;Z_p)\neq 0$, $\Delta\mathfrak{P}^1H^{n+2(p^2-1)(p-1)-1}(K;Z_p)\neq 0$ and the cell $e^{n+2(p^2-1)(p-1)-1}$ is attached to S^n by a representative of β_1^p.

It follows immediately the following

Corollary. $\alpha_1\beta_1^p=0$.

This shows that, in the Adams' spectral sequence computed by May [2], the differential cancells h_0b^p with b_1, or equivalently, the element γ of ${}_pG_{2p^2(p-1)-2}$ does not exist and should be cancelled with $\alpha_1\beta_1^p$. Then the corrected results for ${}_pG_k$ are stated as follows:

Proposition 1. *For $k<2(p^2+2p)(p-1)-4$, ${}_pG_k$ is the direct sum of cyclic groups generated by the following elements of corresponding degree k:*

$\alpha_i(1\leq i<p^2+2p,\ i\not\equiv 0(\mathrm{mod}\ p))$, $\alpha'_{jp}(1\leq j<p+2,\ j\not\equiv 0(\mathrm{mod\ p}))$,
α''_{p^2}, $\beta_1^r(1\leq r<p+3)$, $\alpha_1\beta_1^r(1\leq r<p)$,
$\beta_1^r\beta_s$, $\alpha_1\beta_1^r\beta_s(0\leq r,\ 2\leq s<p,\ r+s<p+2)$, $\beta_2\beta_{p-1}$, $\alpha_1\beta_2\beta_{p-1}$,
$\varepsilon_i(1\leq i<p)$, $\alpha_1\varepsilon_i(1\leq i<p-2)$, ε', $\beta_1\varepsilon'$, φ,

where $\deg(\alpha_i)=2i(p-1)-1$, $\deg(\alpha'_{jp})=2jp(p-1)-1$, $\deg(\alpha''_{p^2})=2p^2(p-1)-1$, $\deg(\beta_s)=2(sp+s-1)(p-1)-2$, $\deg(\varepsilon_i)=2(p^2+i)(p-1)-2$, $\deg(\varepsilon')=2(p^2+1)(p-1)-3$, $\deg(\varphi)=2(p^2+p)(p-1)-3$. *The orders of α'_{jp}, α''_{p^2}* and *φ are p^2, p^3, and p^2 respectively, and the other generators are of order p.* We mention that ε' corresponds to $\alpha_1\gamma$ in [2], $\alpha_1\varepsilon_i$ to $\alpha_{i+1}\gamma$, and $\beta_1\varepsilon'$ to $\alpha_1\beta_1\gamma$. The following representations of new generators are given:

$$\varepsilon'=\{\beta_1^p,\alpha_1,\alpha_1\},\quad \varepsilon_1=\{\alpha_1,\beta_1^p,p\iota,\alpha_1\},$$
$$\varepsilon_{i+1}=\{\varepsilon_i,p\iota,\alpha_1\},\quad 1\leq i<p-1,\quad \varphi\in\{\varepsilon_{p-2},\alpha_1,\alpha_1\}.$$

Among several relations, we remark the following ones:

$p\cdot\varphi=\varepsilon_{p-1}\alpha_1$,

if $p=3$, $\alpha_1\varepsilon'=\beta_1^4$, $\beta_1^2\varepsilon=0$, and $\beta_1^6=0$ (if $p>3$, $\alpha_1\varepsilon'=0$).

Then the first one of our unsolved problem is whether $\beta_1^p\beta_2=0$?

The results of the proposition can be computed from the unstable groups, by use of the results [3] on iterated suspensions, and the groups ${}_p\pi_{n+k}(S^n)$ are completely determined for $k<2(p^2+2p)(p-1)-4$. The details are too long to describe here. Some parts of the results will be noted in the last section. In a mineographed note, Gray has also computed unstable groups for the case $p=3$, using a relations $\{\alpha_1, p\iota, \beta_2\}=0$. But, our results imply $\{\alpha_1, p\iota, \beta_2\}=\pm\beta_1^3$, and his results will be corrected and coincide with ours if one uses the new relation.

2. The proof of the theorem is based on the following two lemmas. We denote by $\varDelta_p(x_i)$ an algebra over Z_p which has a Z_p-basis $x_{i_1}^{a_1}\cdots x_{i_j}^{a_j}$, $0\le a_1,\cdots,a_j<p$. For example, as the dual of $H_*(\Omega S^{2n+1};Z_p)=Z_p[a]$ we have $H^*(\Omega S^{2n+1};Z_p)=\varDelta_p(a_i)$, $\deg(a_i)=2np^{i-1}$, $i=1,2,\cdots$, and $H^*(\Omega^2S^{2n+1};Z_p)=\varLambda(a,\varDelta b_1,\varDelta c_1,\cdots)\otimes\varDelta_p(b_1,b_2,\cdots,c_1,\cdots)$, where $\varDelta=\delta/p$, $\deg(b_1)=2np-2$, $\deg(b_2)=2np^2-2p$, $\deg(c_1)=2np^2-2$. Next, consider a $(2m+1)$-sphere bundle $B_m(p)$ over $S^{2m+2p-1}$ having α_1 as its characteristic class. The loop-space $\Omega B_m(p)$ has the cohomology ring $H^*(\Omega B_m(p);Z_p)=\varDelta_p(a_i,b_i)$, where $\deg(a_i)=2mp^{i-1}$, $\deg(b_i)=2(m+p-1)p^{i-1}$, and $\mathfrak{P}^1a_1=b_1$. Then we have:

Lemma 1. *In $H^*(\Omega B_m(p);Z_p)$, the relation $\mathfrak{P}^{p^{i-1}}a_i=b_i$ holds for sufficiently large m (e.g., $m>p^{i-1}(p-1)$).*

Lemma 2. *In $H^*(\Omega^2S^{2n+1};Z_p)$, we have $\mathfrak{P}^1b_2\neq0$.*

The first lemma is proved by use of p-fold iterated loop multiplication $\mu_p:(\Omega B_m(p))^p\to B_m(p)$. $\mu_p^*(a_i)$, $i>1$, has a term $a_{i-1}\otimes\cdots\otimes a_{i-1}$. By the naturality of $\mathfrak{P}^{p^{i-1}}$ and by Cartan's formula, we have, under inductive assumption, that $\mu_p^*(\mathfrak{P}^{p^{i-1}}a_i)$ and $\mu_p^*(b_i)$ have the same term $b_{i-1}\otimes\cdots\otimes b_{i-1}$. Then the first lemma is proved.

To prove the second lemma, we consider a fibre Q_2^{2n-1} of a fibering equivalent to the canonical inclusion $S^{2n-1}\subset\Omega^2S^{2n+1}$. Then $H^*(Q_2^{2n-1};Z_p)=\varLambda(b_1',\varDelta d',c_1',\cdots)\otimes\varDelta_p(\varDelta b_1',d',e,\varDelta c_1',\cdots)$, where the transgression τ carries $\tau(b_1')=b_1$, $\tau(\varDelta d')=b_2$, $\tau(c_1')=c_1$, and $\deg(e)=2np^2-2p$. Thus it is sufficient to prove $\varDelta\mathfrak{P}^1\varDelta d'=\varDelta c_1'$ in Q_2^{2n-1}. In § 9 of [3], we have a sequence of fiberings:

$$\Omega Q_n\xrightarrow{d}Q_n'\longrightarrow Q_2^{2n-1}\longrightarrow Q_n$$

with natural isomorphisms $H^*(Q_n;Z_p)\approx H^*(\Omega^3S^{2np+1};Z_p)$, $H(\Omega Q_n;Z_p)\approx H^*(\Omega^4S^{2np+1},Z_p)$, $H^*(Q_n';Z_p)\approx H^*(\Omega^2S^{2np-1};Z_p)$. Furthermore, Lemma 9.2 of [3] indicates that the functional $\varDelta\mathfrak{P}^1\varDelta$-operation for d is not trivial, *i.e.*, $\varDelta\mathfrak{P}^1\varDelta(d')\neq0$ in Q_2^{2n-1}. This prove the second Lemma.

Proof of the theorem. Take m sufficiently large and $n=m+p-1$.

Consider the following sequence of fiberings:

$$\Omega^2 S^{2m+2p-1} \xrightarrow{\Omega f} \Omega S^{2m+1} \longrightarrow \Omega B_m(p) \longrightarrow \Omega S^{2m+2p-1} \xrightarrow{f} S^{2m+1}.$$

Lemma 1 implies that the functional $\mathfrak{P}^{p^i}$-operation associated with the mapping Ω^f is not trivial on $H^{2mp^i}(\Omega S^{2m+1}; Z_p)$. ΩS^{2m+1} is homotopy equivalent to the reduced product complex $S^{2m}_\infty = \Sigma_{i\geq 0} e^{2mi}$. For $\Omega^2 S^{2m+2p-1}$, we choose a CW-complex L and a mapping $g: L \longrightarrow \Omega^2 S^{2m+2p-1}$ such that L consists of cells corresponding to each Z_p-basis of $H^*(\Omega^2 S^{2m+2p-1}; Z_p)$ and that g induces an isomorphism of $H^*(\ \ ; Z_p)$. L has a form

$$S^a \cup e^b \cup e^{b+1} \cup \cdots \cup e^{c-2p} \cup e^c \cup e^{c+1} \cup e^{c-2p+1} \cup \cdots$$

for $a = 2m+2p-3$, $b = 2mp+2p(p-1)-2$, $c = 2mp^2+2p^2(p-1)-2$. Consider a cellular map $h: L \longrightarrow S^{2m}$ homotopic to the composition $\Omega f \circ g$. Clearly, $h \mid S^a$ represents α_1. By smashing subcomplexes, h defines $h_1: L^{b+1}/L^a \longrightarrow S^{2mp}$. Since the functional $\mathfrak{P}^p$-operation associated with Ωf, hence with h and h_1, is not trivial, we have $\mathfrak{P}^p \neq 0$ in a mapping cone $S^{2mp} \cup e^{b+1} \cup e^{b+2}$ of h_1. By the triviality of mod p Hopf invariant, $h_1 \mid S^b$ represents β_1, up to non-zero coefficient. Similarly, we have a mapping $h_2: S^{c-2p+2} \cup e^c \cup e^{c+1} \longrightarrow S^{2mp^2}$ such that the mapping cone of h_2, say K, satisifies $\mathfrak{P}^{p^2} \neq 0$. From Lemma 2, the fact $\Delta \mathfrak{P}^1 H^{c-2p+3}(K; Z_p) \neq 0$ follows. Finally, using the H-structure of the mapping Ωf, we can prove easily that $h_2 \mid S^{c-2p+2}$ represents β_1^p. This completes the proof of the theorem.

3. For each element γ of G_k, we define *the unstability* $u(\gamma)$ of γ by

$$u(\gamma) = \text{Min}\,\{n \mid \gamma \in \text{image of } S^\infty: \pi_{n+k}(S^n) \longrightarrow G_k\}.$$

Proposition 2. *Except β_1, each generators, say $\gamma \in {}_pG_k$, listed in Proposition 1 is the S^∞-image of an element $\gamma' \in \pi_{u(\gamma)+k}(S^{u(\gamma)})$ such that γ and γ' have the same order. The same is true for $\gamma = p\alpha'_{jp}$, $p\alpha''_{p^2}$, $p^2\alpha''_{p^2}$, and $p\varphi$.*

This proposition allows us the following direct sum decomposition:

$${}_p\pi_{2m+1+k}(S^{2m+1}) = S(m, k) + U(m, k)$$

for

$$k \neq 2p(p-1)-2$$

and

$$k < 2(p^2+2p)(p-1)-4,$$

where $S(m, k)$ is mapped monomorphically into ${}_pG_k$ by S^∞, and $U(m, k) = \text{Ker}\, S^\infty$. As an analogy of Theorem 11.1 of [3], it is possible to classify the generators of $U(m, k)$, but we need much more types. For example, the composition $\alpha_1\beta_1^p$ gives a quite long series of unstable elements $(1 \leq m < p^2-2)$. Here, we note only the unstability of new stable generators:

$u(\alpha''_{p^2})=7, \quad u(\varepsilon')=2(p-2)p+1, \quad u(\varepsilon_i)=2(p-i)p+3 \quad (1\leq i<p-1),$
$u(\alpha_1\varepsilon_i)=2(p-i-2)p+1 \quad (1\leq i<p-2),$

$$u(\varepsilon_{p-1})=\begin{cases}2p+3 & (p>3)\\ 11 & (p=3),\end{cases} \quad u(\beta_1^{p+1})=\begin{cases}2p-1 & (p>3)\\ 3 & (p=3),\end{cases} \quad u(\beta_1^{p+2})=2p-3,$$

$$u(\varphi)=\begin{cases}7 & (p>3)\\ 9 & (p=3),\end{cases} \quad u(p\cdot\varphi)=\begin{cases}5 & (p>3)\\ 7 & (p=3),\end{cases}$$

$u(\beta_2\beta_{p-1})=u(\beta_1^r\beta_s)=2p-1, \quad u(\alpha_1\beta_2\beta_{p-1})=u(\alpha_1\beta_1^r\beta_s)=3 \quad (r>0).$

References

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