5. *A Remark on the Contraction Principle*

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(Comm. by Kinjirô KUNUGI, M.J.A., Jan. 12, 1968)

In his paper [1], E. Dubinsky states the following fixed point theorem:

Let X_0 be an open neighborhood of 0 in a complete locally convex space E, and let f be a mapping of x_0+X_0, where $x_0 \in E$, into E satisfying the condition that there exist a non-empty closed bounded convex subset B of X_0 and a non-negative real number $k<1$ such that

$$x, y \in x_0+X_0 \text{ and } x-y \in \lambda B \text{ imply } f(x)-f(y) \in \lambda kB.$$

Then if $f(x_0)-x_0 \in (1-k)B$, f has a unique fixed point in x_0+B.

The proof is, in a sense, analogous to that of the well-known Banach contraction theorem, and so it will be natural to ask the relation between these two theorems. The purpose of this note is to clarify the positions of these theorems. That is we shall state a basic theorem (Theorem 1 below) from which these theorems follow, and we shall give a slight generalization of the theorem of Dubinsky (Theorem 2).

The vector spaces we shall be concerned with in this note are over the real number field R or the complex number field. We employ the following notations: $[0, \alpha] \equiv \{\xi \in R; 0 \leq \xi \leq \alpha\}$ and $[0, \alpha) \equiv \{\xi \in R; 0 \leq \xi < \alpha\}$ where α is a positive real number.

1. A triple $\langle X, D, d \rangle$ of a set X, a subset D of $X \times X$ and a non-negative real valued function d defined on D is called a *premetric space* (and d a *premetric* for X with domain D) if the following two conditions are satisfied:

(P 1) For every $x \in X$, $(x, x) \in D$, and $d(x, x)=0$.

(P 2) If $(x, y), (y, z) \in D$, then $(x, z) \in D$ and

$$d(x, z) \leq d(x, y)+d(y, z).$$

Let $\langle X, D, d \rangle$ be a premetric space. If M is a subset of X, then $\langle M, D \cap (M \times M), d|_{D \cap (M \times M)} \rangle$ is also a premetric space, where $d|_{D \cap (M \times M)}$ denotes the restriction of d to $D \cap (M \times M)$; we shall call it a *subspace* of $\langle X, D, d \rangle$ and denote simply by M.

If d is a premetric for a set X with domain D, then by setting $d^*(x, y)=d(y, x)$ for every $(y, x) \in D$, a premetric d^* for X with domain $\{(x, y); (y, x) \in D\}$ is obtained; we shall call d^* the *dual premetric* of d.

A sequence $\{x_n\}$ in a premetric space $\langle X, D, d \rangle$ is *r-convergent* to $x \in X$ if $(x, x_n) \in D$ for every n, and if there exists, for each $\varepsilon < 0$, a positive integer n_0 such that $d(x, x_n) < \varepsilon$ whenever $n \geq n_0$. A sequence is *l-convergent* to x if it is r-convergent to x relative to the dual premetric of d. We say that a premetric space $\langle X, D, d\rangle$ is *r-separated* if every sequence in X is r-convergent to at most one point of X. If $\langle X, D, d\rangle$ is an r-separated premetric space, then as can be readily seen $d(x, y)=0$ implies $x=y$. It is clear that each subspace of an r-separated premetric space is r-separated. A sequence $\{x_n\}$ in a premetric space $\langle X, D, d\rangle$ is an *r-Cauchy sequence* if $(x_m, x_n)\in D$ for $m>n$, and if for each $\varepsilon>0$, there exists a positive integer n_0 such that $m\geq n\geq n_0$ implies $d(x_m, x_n)<\varepsilon$. An r-convergent sequence needs not be an r-Cauchy sequence. A premetric space $\langle X, D, d\rangle$ is *r-complete* if every r-Cauchy sequence in it is r-convergent.

Dually the l-separatedness, the l-Cauchy sequences, and the l-completeness are defined. In what follows, without Lemma 2, we confine ourselves to the case where "r" is prefixed. However, every result may be translated by the duality to the other case.

Let X be a set, and let D a subset of $X\times X$. For each $a\in X$, we denote by $D(a)$ the set of all $x\in X$ with $(x, a)\in D$.

We conclude this section by the following lemma which may be verified easily.

Lemma 1. *If $\langle X, D, d\rangle$ is an r-complete premetric space, then for each positive real number α and $a\in X$, the subspace $\{x\in D(a);\ d(x, a)\leq\alpha\}$ is r-complete.*

2. Let $\langle X, D, d\rangle$ be a premetric space, and let $k\in[0, 1)$. A mapping f of X into itself is called a *k-contraction* if $(x, y)\in D$ implies $(f(x), f(y))\in D$ and $d(f(x), f(y))\leq kd(x, y)$.

It is easy to see that, in an r-separated premetric space $\langle X, D, d\rangle$, if $x, y\in X$ are fixed points of a k-contraction with $k\in[0, 1)$ and if $(x, y)\in D$, then $x=y$.

Now we can state the Banach contraction theorem for premetric spaces:

Theorem 1. *Let $\langle X, D, d\rangle$ be an r-separated premetric space, and f a k-contraction of X into itself with $k\in[0, 1)$. If there exists a point $a\in X$ such that $(f(a), a)\in D$, and if the subspace $M=\{x\in D(a);\ d(x, a)\leq(1-k)^{-1}d(f(a), a)\}$ is r-complete, then there exists a unique $x\in D(a)$ such that $f(x)=x$; moreover $x\in M$ and the sequence $\{f^n(a)\}$ is r-convergent to x.*

Proof. Since $(f(a), a)\in D$, each pair $(f^{n+1}(a), f^n(a))$, $n=1, 2, \cdots$, does belong to D. Hence by induction, we can show that, for each positive integer n, $(f^n(a), a)\in D$ and

$$d(f^n(a), a)\leq d(f^n(a), f^{n-1}(a))+d(f^{n-1}(a), f^{n-2}(a))+\cdots+d(f(a), a)$$
$$\leq(1-k)^{-1}d(f(a), a),$$

and so we have, for every positive integer m,

$$d(f^{n+m}(a), f^n(a))\leq k^n d(f^m(a), a)\leq k^n(1-k)^{-1}d(f(a), a).$$

This inequality shows that $\{f^n(a)\}$ is an r-Cauchy sequence in the subspace M. Consequently, it is r-convergent to a point $x \in M$, that is, for each $\varepsilon > 0$, there is a positive integer n_0 such that $n \geq n_0$ implies $d(x, f^n(a)) < \varepsilon$, and hence we have

$$d(f(x), f^n(a)) \leq kd(x, f^{n-1}(a)) < \varepsilon \qquad \text{for every } n \geq n_0 + 1.$$

Thus the sequence $\{f^n(a)\}$ is also r-convergent to $f(x)$. Since the premetric space is r-separated, this implies $f(x) = x$. Now if $y \in D(a)$ and $f(y) = y$, then the inequality $d(y, f^n(a)) \leq k^n d(y, a)$ shows that $\{f^n(a)\}$ is r-convergent to y, and so we have $y = x$ by the same reason. This completes the proof.

3. A subset B of a vector space E is said to be *star-shaped* if $\lambda B \subset B$ for every $\lambda \in [0, 1]$. For each subset B of E, we denote by $S(B)$ the union of all λB with $\lambda \in [0, 1]$. If B is convex, then $S(B)$ coincides with the convex hull of the set $\{0\} \cup B$. A subset B of E is *circled* if $|\lambda| \leq 1$ implies $\lambda B \subset B$.

Lemma 2. *Let B be a non-empty bounded star-shaped convex subset of a Hausdorff topological vector space E. Denote by D the set of all $(x, y) \in E \times E$ such that $x - y \in \lambda B$ for some $\lambda > 0$, and put, for each $(x, y) \in D$,*

$$d(x, y) = \inf \{\lambda > 0;\ x - y \in \lambda B\}.$$

Then $\langle E, D, d \rangle$ is an r-separated and l-separated premetric space.

Proof. The condition (P 1) is obviously satisfied. To verify the condition (P 2), let $(x, y), (y, z) \in D$. Then for some $\lambda > 0$ and $\mu > 0$, we have $x - y \in \lambda B$ and $y - z \in \mu B$, which imply $x - z \in (\lambda + \mu)B$. This shows that $(x, z) \in D$ and $d(x, z) \leq d(x, y) + d(y, z)$. Thus $\langle E, D, d \rangle$ is a premetric space. Now let $\{x_n\}$ be a sequence in E which is r-convergent (resp. l-convergent) to two points $x, y \in E$ at the same time. For each neighborhood U of 0 in E, we can find a circled neighborhood V of 0 in E such that $V + V \subset U$. Take an $\varepsilon > 0$ with $\varepsilon B \subset V$. Then for sufficiently large n, both $x - x_n$ and $y - x_n$ (resp. $x_n - x$ and $x_n - y$) belong to εB. Since V is circled, $-y + x_n$ (resp. $x - x_n$) belongs to V, and hence we have $x - y = x - x_n + x_n - y \in V + V \subset U$. Since E is Hausdorff, it follows that $x = y$. Therefore $\langle E, D, d \rangle$ is r-separated and l-separated.

Lemma 3. *If B is a sequentially complete bounded subset of a Hausdorff topological vector space E, then $S(B)$ is sequentially complete.*

Proof. Denote by $e(B)$ the set of all $x \in B$ such that $\lambda x \overline{\in} B$ for every $\lambda > 1$. We shall show first that, for each $x \in B$, there exists a $\lambda_0 \geq 1$ with $\lambda_0 x \in e(B)$. Let $\lambda_0 = \sup \{\lambda;\ \lambda x \in B\}$. Then $\lambda_0 \geq 1$, and we can find a sequence $\{\lambda_n\}$ in $\{\lambda;\ \lambda x \in B\}$ which converges to λ_0. Since the sequence $\{\lambda_n x\}$ converges to $\lambda_0 x$, and since B is sequentially complete, $\lambda_0 x$ belongs to B. In addition, $\lambda > 1$ implies $\lambda\lambda_0 x \overline{\in} B$. Thus

$\lambda_0 x \in e(B)$. Now let $\{a_n\}$ be a Cauchy sequence in $S(B)$. Then for each positive integer n, we can find a $\lambda_n \in [0, 1]$ and an $x_n \in e(B)$ such that $a_n = \lambda_n x_n$. The sequence $\{\lambda_n\}$ contains a convergent subsequence. Since $\{a_n\}$ is a Cauchy sequence, it suffices to show that a subsequence of $\{a_n\}$ converges to an element of $S(B)$, and so we may assume without loss of generality that the sequence $\{\lambda_n\}$ converges to a number $\lambda \in [0, 1]$. Let U be an arbitrary neighborhood of 0 in E. Then there exists a circled neighborhood V of 0 in E such that $V + V \subset U$. Take a positive real number β with $\beta B \subset V$. If $\lambda = 0$, then we can find a positive integer n_0 for which we have $\lambda_m < \beta$ for every $m \geq n_0$; consequently we have $a_m = \lambda_m x_m \in U$, which shows that $\{a_n\}$ converges to $0 \in S(B)$. Now consider the case where $\lambda \neq 0$. We can assume that $\lambda_n \neq 0$ for every n. Let $0 < \varepsilon < \min\{\lambda, 2\beta\lambda/(2+\beta)\}$. Then there exists a positive integer n_0 such that

$$|\lambda_m - \lambda| < \varepsilon/2 \text{ and } \lambda_m x_m - \lambda_n x_n \in (\lambda - \varepsilon)V \qquad \text{for every } m, n \geq n_0.$$

We have, for every $m, n \geq n_0$,

$$\left|\frac{\lambda_n}{\lambda_m} - 1\right| < \frac{\varepsilon}{\lambda_m} < \frac{\varepsilon}{\lambda - \dfrac{\varepsilon}{2}} < \beta,$$

and hence

$$\left(\frac{\lambda_n}{\lambda_m} - 1\right)x_n \in \left(\frac{\lambda_n}{\lambda_m} - 1\right)B \subset V.$$

On the other hand, since $0 < \lambda - \varepsilon < \lambda_m$, we have

$$x_m - \frac{\lambda_n}{\lambda_m} x_n \in \left(\frac{\lambda - \varepsilon}{\lambda_m}\right)V \subset V \qquad \text{for every } m, n \geq n_0.$$

Therefore, if $m, n \geq n_0$, then we have

$$x_m - x_n = x_m - \frac{\lambda_n}{\lambda_m} x_n + \left(\frac{\lambda_n}{\lambda_m} - 1\right)x_n \in V + V \subset U.$$

It follows that $\{x_n\}$ is a Cauchy sequence in B, and so it converges to an element $a \in B$. Since the sequence $\{\lambda_n\}$ converges to λ, the sequence $\{a_n\}$ converges to λa. This completes the proof.

Now we have the following theorem.

Theorem 2. *Let X_0 be a subset of a Hausdorff topological vector space E, and f a mapping of X_0 into E satisfying the condition that there exists a non-empty sequentially complete bounded convex subset B of E and a $k \in [0, 1)$ such that*

$$x, y \in X_0 \text{ and } x - y \in \lambda B (\lambda \geq 0) \text{ imply } f(x) - f(y) \in \lambda k B.$$

If there exists an element $a \in E$ such that $a + S(B) \subset X_0$ and $f(a) - a \in \alpha B$, where $\alpha \in [0, 1-k]$, then f has a unique fixed point in $a + \alpha(1-k)^{-1}B$; moreover the sequence $\{f^n(a)\}$ converges to the fixed point.

Proof. Let us denote by X the set of all $x \in X_0$ such that $f^n(x) \in X_0$ for every positive integer n. Then, the restriction of f

to X is a mapping of X into X. We shall show that $a+S(B)$ is contained in X. To this end, take an arbitrary element x of $a+S(B)$. Then for some $\lambda\in[0,1]$, the element $x-a$ belongs to λB, and so we have $f^n(x)-f^n(a)\in\lambda k^n B$ for every positive integer n. On the other hand, since $f(a)-a\in\alpha B$, we have $f^n(a)-f^{n-1}(a)\in\alpha k^{n-1}B$ for every $n\geq 2$. Consequently we have, for every positive integer n,

$$(*)\qquad \begin{aligned} f^n(x)-a&=f^n(x)-f^n(a)+f^n(a)-f^{n-1}(a)+\cdots+f(a)-a\\ &\in\lambda k^n B+\alpha k^{n-1}B+\cdots+\alpha B\\ &\subset\Big(\lambda k^n+\alpha\frac{1-k^n}{1-k}\Big)B=B', \end{aligned}$$

where the set B' is contained in $S(B)$, because of the relation

$$0\leq\Big(\lambda k^n+\alpha\frac{1-k^n}{1-k}\Big)\leq 1-(1-\lambda)k^n\leq 1.$$

Thus we have $f^n(x)\in a+S(B)\subset X_0$ for every positive integer n, which establishes that $a+S(B)\subset X$. Therefore, it suffices to prove the theorem under the hypothesis that the mapping f is of X_0 into X_0.

Now consider the set D of all $(x,y)\in E\times E$ such that $x-y\in\lambda S(B)$ for some $\lambda>0$, and a function d on D defined by

$$d(x,y)=\inf\{\lambda>0;\ x-y\in\lambda S(B)\}.$$

Since the set $S(B)$ is bounded star-shaped convex subset of E, by virtu of Lemma 2, $\langle E,D,d\rangle$ is an r-separated premetric space, and hence so is the subspace X_0. It is clear that f is a k-contraction of X_0 into X_0. Moreover the hypothesis of the theorem shows that $(f(a),a)$ belongs to the set D. Therefore if we prove that the set $M=\{x\in D(a);\ d(x,a)\leq(1-k)^{-1}d(f(a),a)\}$ is r-complete, then it follows from Theorem 1 immediately that f has a unique fixed point x_0 in $D(a)$ to which the sequence $\{f^n(a)\}$ is r-convergent. Then, for each neighborhood U of 0 in E, a positive real number λ exists with $\lambda S(B)\subset U$; and hence we can find a positive integer n_0 such that

$$x_0-f^n(a)\in\lambda S(B)\subset U\qquad\text{for every } n\geq n_0.$$

This shows that the sequence $\{f^n(a)\}$ converges to x_0 relative to the original topology of E.

We shall proceed to prove that the set M is r-complete. Since M is contained in the set $a+S(B)$, it suffices, by Lemma 1, to show that $a+S(B)$ is r-complete. Let $\{x_n\}$ be an r-Cauchy sequence in $a+S(B)$. Then, for each neighborhood U of 0 in E, there is a $\lambda>0$ with $\lambda S(B)\subset U$. Hence we can find a positive integer n_0 such that $m\geq n\geq n_0$ implies $x_m-x_n\in\lambda S(B)\subset U$. Thus the sequence $\{x_n\}$ is a Cauchy sequence in the sequentially complete subset $a+S(B)$. Consequently, $\{x_n\}$ converges to an element $x\in a+S(B)$. On the other hand, for each $\varepsilon>0$, there exists a positive integer n_0 such that $m\geq n\geq n_0$ implies $x_m-x_n\in\varepsilon S(B)$. It follows that $\{x_m;\ m\geq n\}$ is a Cauchy sequence in $x_n+\varepsilon S(B)$ for every $n\geq n_0$, and so x does belong to $x_n+\varepsilon S(B)$ for every $n\geq n_0$. Therefore the sequence $\{x_n\}$ is r-convergent to x.

It remains only to prove that the fixed point x_0 belongs to $a+\alpha(1-k)^{-1}B$. It is sufficient to consider the case where $\alpha\neq 0$. By the relation $(*)$, we have

$$f^n(a)-a\in(k^{n-1}+k^{n-2}+\cdots+1)\alpha B=\frac{1-k^n}{1-k}\alpha B$$

for every positive integer n.

Now the sequence $\left\{\frac{1-k}{(1-k^n)\alpha}\right\}$ converges to $(1-k)/\alpha$, and $\{f^n(a)\}$ converges to the fixed point x_0. Hence the sequence $\left\{\frac{1-k}{(1-k^n)\alpha}(f^n(a)-a)\right\}$ in B converges to $((1-k)/\alpha)(x_0-a)$, and so we obtain the desired conclusion, since B is sequentially complete.

Reference

[1] E. Dubinsky: Fixed points in non-normed spaces. Ann. Acad. Sci. Fennicae A. I., **331** (1963).